An algebraic-geometric view on tensor decomposition problems

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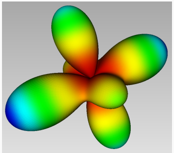


Journée scientifique du pôle AM2I : "Introduction à la géométrie des tenseurs" - Nancy - 3rd November 2025

Matrices

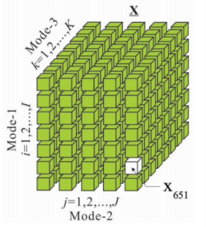
$M \in \mathbb{K}^{n_1 \times n_2} = \mathbb{K}^{n_1} \otimes \mathbb{K}^{n_2}$ is of **rank** r iff there exist $U \in \mathbb{K}^{n_1 \times n_1}$, $V \in \mathbb{K}^{n_2 \times n_2}$ invertible and Σ_r diagonal invertible s.t.

$$M = U \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} V^t$$

- Σ_r not unique
- $\Sigma_r = I_r$ for some U, V.
- U, V unitary $\Rightarrow$ Singular Value Decomposition
- U, V are **eigenvectors** of $M\,M^t$ (resp. $M^t M$)
- Best low rank approximation from truncated SVD

Multilinear tensors of $\mathbb{K} = \mathbb{R}, \mathbb{C}, \ldots$

A tri-linear tensor $T \in \mathbb{K}^{n_1 \times n_2 \times n_3} = \mathbb{K}^{n_1} \otimes \mathbb{K}^{n_2} \otimes \mathbb{K}^{n_3}$



Decomposition of a trilinear tensor

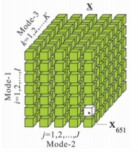
$$T = \sum_{j=1}^{r} U_j \otimes V_j \otimes W_j \text{ with } U_j \in \mathbb{K}^{n_1}, V_j \in \mathbb{K}^{n_2}, W_j \in \mathbb{K}^{n_3}$$

with r minimal.

Coefficent-wise: $T_{i_1,i_2,i_3} = \sum_{j=1}^{r} U_{i_1,j} V_{i_2,j} W_{i_3,j}$

Decomposition and diagonalisation

$T \in \mathbb{K}^{n_1} \otimes \mathbb{K}^{n_2} \otimes \mathbb{K}^{n_3} \quad \equiv \quad [T_{[i]}]_{i=1}^{n_3}$ pencil of n_3 matrices of size $n_1 \times n_2$.



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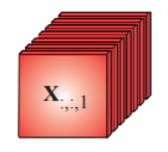

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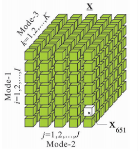

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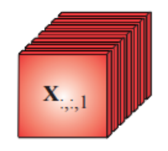

For $T \in \mathbb{K}^{\mathsf{r}} \otimes \mathbb{K}^{\mathsf{r}} \otimes \mathbb{K}^{n_3}$,

$$T = \sum_{j=1}^{\mathsf{r}} U_j \otimes V_j \otimes W_j \text{ with } U, V \in \mathbb{K}^{\mathsf{r} \times \mathsf{r}}, W \in \mathbb{K}^{n_3 \times \mathsf{r}}$$

iff $\mathrm{T}_{[i]} = U \operatorname{diag}(W_{i,1}, \ldots, W_{i,\mathsf{r}}) V^t \quad i \in 1 : n_3$

If $T_{[1]}$ inv., $\quad U =$ matrix of **common eigenvectors** of $M_i = T_{[i]} T_{[1]}^{-1}$

$V^{-t} =$ matrix of **common eigenvectors** of $M_i' = T_{[1]}^{-1} T_{[i]}$.

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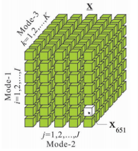

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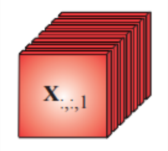

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$$T = \sum_{j=1}^{r} U_j \otimes V_j \otimes W_j \text{ with } U, V \in \mathbb{K}^{r \times r}, W \in \mathbb{K}^{n_3 \times r}$$

iff $\quad T_{[i]} = U \operatorname{diag}(W_{i,1}, \ldots, W_{i,r}) V^t \quad i \in 1 : n_3$

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$\qquad V^{-t} =$ matrix of **common eigenvectors** of $M_i' = T_{[1]}^{-1}\, T_{[i]}$.

Decomposition (when $T_0 = \sum_i l_i T_{[i]}$ invertible):

- Compute the common eigenvectors U of $M_i = T_{[i]} T_0^{-1}$ for $T_0 = \sum_i l_i T_{[i]}$;
- Deduce the common eigenvectors $V^{-t}\Sigma_0 = T_0^{-1} U$ of $M_i' = T_0^{-1} T_{[i]}$ and $\mathrm{diag}(W_{i,1}, \ldots, W_{i,r}) = U T_{[i]} V^{-t}$;

Known as *Jennrich* algorithm (1970?).

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☞ **We associate to the decomposition of T an Artinian algebra:**

- The matrices $(M_i)_{i\in[n]}$ of size r are commuting.
- Let $R = \mathbb{K}[_1, \ldots, x_n] = \mathbb{K}[\boldsymbol{x}]$ be the ring of polynomials in the variables $x_1, \ldots, x_n$, $v \in \mathbb{K}^n$ and

$$I = \{p(x_1, \ldots, x_n) \in R : p(M_1, \ldots, M_n)(v) = 0\}$$

- I is an ideal of R (vector space of $\mathbb{K}$ stable by mult. by $q \in R$); $\mathcal{A} = R/I$ the quotient algebra, is of dimension $\leq r$ (i.e. *Artinian* algebra).

Sequences of values

Given a sequence of values

$$\varphi_0, \varphi_1, \ldots, \varphi_s \in \mathbb{C},$$

☞ Find/guess the values of φ_n for all $n \in \mathbb{N}$.

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☞ Find $r \in \mathbb{N}, \omega_i, \xi_i \in \mathbb{C}$ such that $\varphi_n = \sum_1^r \omega_i \, \xi_i{}^n$, for all $n \in \mathbb{N}$.

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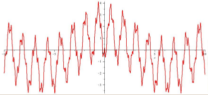
Example: 0, 1, 1, 2, 3, 5, 8, 13, ...

Solution:

- Find a recurrence relation valid for the first terms: $\varphi_{k+2} - \varphi_{k+1} - \varphi_k = 0$.
- Find the roots $\xi_1 = \frac{1+\sqrt{5}}{2}$, $\xi_2 = \frac{1-\sqrt{5}}{2}$ (golden numbers) of the characteristic polynomial: $x^2 - x - 1 = 0$.
- Deduce $\varphi_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$.

Prony's method (1795)



For the signal $f(t) = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}} e^{\zeta_{\mathbf{i}} t}$, $(\omega_i, \zeta_i \in \mathbb{C})$,

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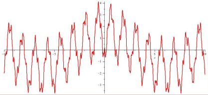

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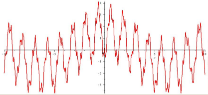

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- Find $p = [p_0, \ldots, p_r]$ with $p_r = 1$ s.t. $\varphi_{k+r} + \varphi_{k+r_1} p_{r-1} + \cdots + \varphi_k p_0 = 0$ by solving:

$$\begin{bmatrix} \varphi_0 & \varphi_1 & \cdots & \varphi_r \\ \varphi_1 & & & \varphi_{r+1} \\ \vdots & & & \vdots \\ \varphi_{r-1} & \cdots & \varphi_{2r-1} & \varphi_{2r-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

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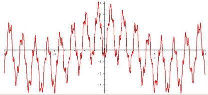

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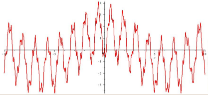

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- Solve the system

$$\begin{bmatrix} 1 & \ldots & \ldots & 1 \\ \xi_1 & & & \xi_r \\ \vdots & & & \vdots \\ \xi_1^{r-1} & \ldots & \ldots & \xi_r^{r-1} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{\mathrm{r}} \end{bmatrix} = \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \vdots \\ \varphi_{r-1} \end{bmatrix}$$

Since $p_r = 1$, we have $-\varphi_i p_0 - \cdots - \varphi_{i+r-1} p_{r-1} = \varphi_{i+r}$ and

$$\overbrace{\begin{pmatrix} \varphi_0 & \varphi_1 & \cdots & \varphi_{r-1} \\ \varphi_1 & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \varphi_{r-1} & \varphi_r & \cdots & \varphi_{2r-2} \end{pmatrix}}^{H_0} \overbrace{\begin{pmatrix} 0 & & & & -p_0 \\ 1 & \ddots & & & -p_1 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & 0 & \vdots \\ & & & 1 & -p_{r-1} \end{pmatrix}}^{M_x} = \overbrace{\begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_r \\ \varphi_2 & & & \vdots \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \varphi_r & \varphi_{r+1} & \cdots & \varphi_{2r-1} \end{pmatrix}}^{H_1}$$

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$M_x = H_0^{-1} H_1$ compagnon matrix of p

$= \text{matrix of multiplication by x in } \mathcal{A} = \mathbb{K}[\mathrm{x}]/(\mathrm{p})$

in the basis $\{1, x, \ldots, x^{r-1}\}$

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$= $ matrix of multiplication by x in $\mathcal{A} = \mathbb{K}[x]/(p)$

in the basis $\{1, x, \ldots, x^{r-1}\}$

☞ **We associate with φ, the Artinian algebra $\mathcal{A} = \mathbb{K}[x]/(p)$.**

Sequences and symmetric tensors

Given a **sequence of values** $\varphi = (\varphi_0, \ldots, \varphi_d)$, we define the **symmetric tensor** or binary form:

$$F = \sum_{i=0}^{d} \varphi_i \binom{d}{i} x_0^{d-i} x_1^i$$

Using the **apolar product:** for $f = \sum_{|\alpha|=d} f_\alpha \, \mathrm{x}^\alpha$,$g = \sum_{|\alpha|=d} g_\alpha \, \mathrm{x}^\alpha$,

$$\langle f, g \rangle_d = \sum_{|\alpha|=d} f_\alpha \, g_\alpha \binom{d}{\alpha}^{-1}.$$

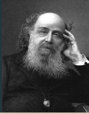
we have $\langle F, g \rangle = \sum_i \varphi_i g_{(d-i,i)}$.

If $\varphi_i = \sum_{j=1}^{r} \omega_i \, \xi_j^i$ then

☞ $F = \sum_{j=1}^{r} \omega_i \, (x_0 + \xi_j x_1)^d$

☞ $\langle F, g \rangle_d = \sum_{j=1}^{r} \omega_i \; g(1, \xi_j)$

Sylvester approach (1851)



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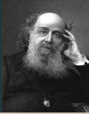

Theorem:

The binary form $T(x_0, x_1) = \sum_{i=0}^{d} t_i \binom{d}{i} x_0^{d-i} x_1^i$ can be decomposed as a sum of r distinct powers of linear forms

$$T = \sum_{k=1}^{r} \omega_k (\alpha_k x_0 + \beta_k x_1)^d$$

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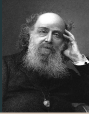

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iff there exists a polynomial $p(x_0, x_1) := p_0 x_0^r + p_1 x_0^{r-1} x_1 + \cdots + p_r x_1^r$ s.t.

$$\begin{bmatrix} t_0 & t_1 & \ldots & t_r \\ t_1 & & & t_{r+1} \\ \vdots & & & \vdots \\ t_{d-r} & \ldots & t_{d-1} & t_d \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

and of the form $p = c \prod_{k=1}^{r} (\beta_k x_0 - \alpha_k x_1)$ with $(\alpha_k : \beta_k)$ distinct.

If $\alpha_k \neq 0$, $\xi_k = \frac{\beta_k}{\alpha_k}$ root of $p(x) = \sum_{i=0}^{r} p_i x^i$ (or generalized eigenvalues of (H_0, H_1)).

Example with Fibonacci sequence $\varphi = (0, 1, 1, 2, 3, 5, 8, 13, \ldots)$, $d = 4$

- $F = \sum_{i=0}^{4} \varphi_i \binom{d}{i} x_0^{d-i} x_1^i = 4x_0^3 x_1 + 6x_0^2 x_1^2 + 8x_0 x_1^3 + 3x_1^4$
- $k = 2, d - k = 2$

$$H_F^{2,2} \quad = \quad (\varphi_{i+j})_{0 \leq i,j \leq 3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

- rank $H_F^{2,2} = 2$
- $H_T^{2,2} = \begin{bmatrix} 1 & 1 \\ \xi_1 & \xi_2 \\ \xi_1^2 & \xi_2^2 \end{bmatrix} \mathrm{diag}(\omega_1, \omega_2) \begin{bmatrix} 1 & \xi_1 & \xi_1^2 \\ 1 & \xi_2 & \xi_2^2 \end{bmatrix}$

  with ξ_i roots of $X^2 - X - 1 = 0$ for $X = \frac{x_1}{x_0}$.

Symmetric tensor decomposition and Waring problem (1770)

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Symmetric tensor decomposition problem:

Given a homogeneous polynomial $F \in \mathcal{S}_d$ of degree d in the variables $\underline{\boldsymbol{x}} = (x_0, x_1, \ldots, x_n)$ with coefficients $\in \mathbb{K}$:

$$F(\overline{\boldsymbol{x}}) = \sum_{|\alpha|=d} F_\alpha \, \underline{\boldsymbol{x}}^\alpha,$$

find a minimal decomposition of F of the form

$$F(\underline{\boldsymbol{x}}) = \sum_{i=1}^{r} \omega_i (\xi_{i,0} x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)^d = \sum_{i=1}^{r} \omega_i \, (\underline{\xi}_i, \underline{\boldsymbol{x}})^d$$

with $\underline{\xi}_i = (\xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,n}) \in \overline{\mathbb{K}}^{n+1}$ spanning disctint lines, $\omega_i \in \overline{\mathbb{K}}$.

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The minimal r in such a decomposition is called the **rank** of T.

Generalized Additive or Iarrobino Decomposition

Generalized Additive Decomposition problem:

find r', $w_i(\underline{\boldsymbol{x}}) \in \mathcal{S}^{k_i}$ for $i = 1, \ldots, r'$ and $\Xi = [\xi_1, \ldots, \xi_{r'}] \in \mathbb{K}^{(n+1)\times r'}$ such that

$$F = \sum_{i=1}^{r'} \omega_i(\underline{\boldsymbol{x}})\, \ell_i(\xi_i, \underline{\boldsymbol{x}})^{d-k_i}$$

with $\ell_i = (\xi_i, \underline{\boldsymbol{x}})$ **not dividing** ω_i and $r = \sum_i \mathrm{rank}_{\ell_i}(\omega_i)$ **minimal**.

Generalized Additive or Iarrobino Decomposition

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Example: For $d > 5$, $F = x_0^{d-1}x_1 + (x_0 + x_1 + 2\,x_2)^{d-2}(x_0 - x_1)^2$ is a GAD with

$$\begin{aligned} r &= \mathrm{rank}_{\ell_1}(x_1) + \mathrm{rank}_{\ell_2}(x_0 - x_1)^2 \quad \text{(defined latter)} \\ &= 2 + 3 = 5 \end{aligned}$$

with $\ell_1 = x_0$, $\ell_2 = x_0 + x_1 + 2x_2$.

Geometric point of view

- $\mathcal{V}_{n+1,d} = \{\omega\,(\xi, \underline{\boldsymbol{x}})^d, \omega \in \mathbb{K}, \xi \in \mathbb{K}^{n+1}, \xi \neq 0\}$ **Veronese** variety
- $\mathcal{T}_{n+1,d} = \{\omega(\underline{\boldsymbol{x}})\,(\xi, \underline{\boldsymbol{x}})^{d-1}, \omega(\underline{\boldsymbol{x}}) \in \mathcal{S}^1, \xi \in \mathbb{K}^{n+1}, \xi \neq 0\}$ **tangential** variety (= points on tangents to $\mathcal{V}_{n+1,d}$).
- $\mathcal{O}^k_{n+1,d} = \{\omega(\underline{\boldsymbol{x}})\,(\xi, \underline{\boldsymbol{x}})^{d-k}, \omega(\underline{\boldsymbol{x}}) \in \mathcal{S}^k, \xi \in \mathbb{K}^{n+1}, \xi \neq 0\}$ **osculating** variety (= points on oscullating linear spaces to $\mathcal{V}_{n+1,d}$).

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The singular locus of $\mathcal{O}^k_{n+1,d}$ is $\mathcal{O}^{k-1}_{n+1,d}$, $\mathcal{V}_{n+1,d} = \mathcal{O}^0_{n+1,d}$ is smooth.

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$$F = \sum_{i=1}^{r'} \omega_i(\underline{\boldsymbol{x}})(\xi_i, \underline{\boldsymbol{x}})^{d-k_i} \text{ is a GAD} \quad \text{iff} \quad F \in \sum_{i=1}^{r'} (\mathcal{O}^{k_i}_{n+1,d})^{\text{smooth}}$$

From sequences to symmetric tensors

- **From multi-index sequences:** $\varphi = (\varphi_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{K}^{\mathbb{N}^n}$ indexed by $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, called **moment sequences**.

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$(\boldsymbol{y}^\alpha)_{\alpha \in \mathbb{N}^n}$ (resp. $(\frac{1}{\alpha!}\boldsymbol{z}^\alpha)_{\alpha \in \mathbb{N}^n}$) dual basis in R^* of the monomial basis $(\boldsymbol{x}^\alpha)_{\alpha \in \mathbb{N}^n}$

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$\varphi(\boldsymbol{y}) = \sum_{\alpha \in \mathbb{N}^n} \varphi_\alpha \boldsymbol{y}^\alpha \in \mathbb{K}[[y_1, \ldots, y_n]]$ $\qquad$ $\varphi(\boldsymbol{z}) = \sum_{\alpha \in \mathbb{N}^n} \varphi_\alpha \frac{\boldsymbol{z}^\alpha}{\alpha!} \in \mathbb{K}[[\mathrm{z}_1, \ldots, \mathrm{z}_n]]$

where $\alpha! = \prod \alpha_i!$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}$.

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- **to symmetric tensors of degree d:**

$$\varphi^{[d]} = \sum_{|\alpha|\le d} \varphi_\alpha \binom{d}{\alpha} x_0^{d-|\alpha|} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathcal{S}^d.$$

From tensors to linear functionals

Apolar product: For $F = \sum_{|\alpha|=d} f_\alpha \, \underline{\boldsymbol{x}}^\alpha$, $F' = \sum_{|\alpha|=d} f'_\alpha \, \underline{\boldsymbol{x}}^\alpha \in \mathcal{S}^d$,

$$\langle F, F' \rangle_d = \sum_{|\alpha|=d} \binom{d}{\alpha}^{-1} f_\alpha \, f'_\alpha.$$

Properties:

- $\langle F, (\boldsymbol{u} \cdot \underline{\boldsymbol{x}})^d \rangle_d = F(\boldsymbol{u})$
- $\langle F, (\boldsymbol{v}_1 \cdot x) \cdots (\boldsymbol{v}_k \cdot \underline{\boldsymbol{x}})(\boldsymbol{u} \cdot \underline{\boldsymbol{x}})^{d-k} \rangle = \frac{(d-k)!}{d!} D_{\boldsymbol{v_1}} \cdots D_{\boldsymbol{v}_k} F(\boldsymbol{u})$

☞ For an **orthonormal** basis $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ of $\mathbb{K}^{n+1}$, $(\sqrt{\binom{d}{\alpha}} \prod_{i=1}^n (\boldsymbol{u}_i \cdot \underline{\boldsymbol{x}})^{\alpha_i})_{|\alpha|=d}$ **orthonormal basis of $\mathcal{S}^d$.**

Definition

For $F \in \mathcal{S}^d$ and $h \in \mathcal{S}^k$,

$$F^* : p \in \mathcal{S}^d \mapsto \langle F, p \rangle \in \mathbb{K} \text{ is a linear functiona} \in \mathcal{S}^{d*}$$

Co-multiplication by $\mathcal{S}$: $q \star F^* : p \in \mathcal{S}^{d-k} \mapsto \langle F, p\, q \rangle$

Decompositions of tensors or series

If $F = \sum_{i=1}^{r} \omega_i \ell_i^d$ with $\ell_i = (x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)$, then

$$F^* = \sum_{i=1}^{r} \omega_i \delta_{\underline{\xi_i} | \mathcal{S}^d} \text{ with } \delta_{\underline{\xi_i}} \textbf{ Dirac} \text{ or } \textbf{evaluation} \text{ at } \underline{\xi}_i = (1, \xi_i)$$

$$F^*(\boldsymbol{z}) = \Big(\sum_{i=1}^{r} \omega_i \boldsymbol{e}_{\underline{\xi}_i}(\underline{\boldsymbol{z}})\Big)^{[d]} \text{ with } \boldsymbol{e}_{\xi}(\underline{\boldsymbol{z}}) = \sum_{\alpha} \xi^{\alpha} \frac{\underline{\boldsymbol{z}}^{\alpha}}{\alpha!}, \bullet^{[d]} = \text{degree } d \text{ part.}$$

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If $F = \sum_{i=1}^{r} \omega_i \ell_i^{d-k_i}$ with $\ell_i = (x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)$, then

$$F^*(\boldsymbol{z}) = \Big(\sum_{i=1}^{r} \omega_i^{d,\ell_i,\boldsymbol{z}}(\boldsymbol{z}) \boldsymbol{e}_{\underline{\xi}_i}(\underline{\boldsymbol{z}})\Big)^{[d]}$$

$$\check{F}(\boldsymbol{z}) = \Big(\sum_{i=1}^{r} \omega_i^{d,\ell_i,\boldsymbol{z}}(\boldsymbol{z}) \boldsymbol{e}_{\xi_i}(\boldsymbol{z})\Big)^{[\leq d]} \text{ setting } x_0 = 1$$

$$\text{with } \omega_i = \sum_{j=0}^{k_i} \omega_{i,j}(\boldsymbol{x})\, \ell_i^{k_i - j} \text{ and } \omega_i^{d,\ell_i,\boldsymbol{x}} = \sum_{j} \frac{(d-j)!}{d!} \omega_{i,j}(\boldsymbol{x})$$

☞ Truncation of a **polynomial-exponential series** in degree $\leq d$.

From polynomial-exponential series to Artinian algebras

Given a series $\varphi = \sum_\alpha \varphi_\alpha \frac{\boldsymbol{z}^\alpha}{\alpha!} \in \mathbb{K}[[\boldsymbol{z}]]$,

- **Annihilator** of φ:

  $$I_\varphi = \mathrm{Ann}(\varphi) = \{p \in \mathbb{K}[\boldsymbol{z}] : p\star\varphi = 0\} = \{p \in \mathbb{K}[\boldsymbol{z}] : \forall q \in \mathbb{K}[\boldsymbol{z}]\ \langle \varphi, p\, q\rangle = 0\}$$

- **Quotient algebra** of φ: $\mathcal{A}_\varphi = \mathbb{K}[\boldsymbol{x}]/I_\varphi$.
- **Hankel** operator of φ: $H_\varphi : p \in \mathbb{K}[\boldsymbol{x}] \mapsto p \star \varphi \in \mathbb{K}[\boldsymbol{x}]^*$

H_φ is also known as a **convolution** operator. When restricted to degrees $\leq (d-c, c)$, its matrix is known as **Catalecticant** or **moment matrix**.

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φ is a **polynomial-exponential** series
iff
$\mathcal{A}_\varphi$ is an **Artinian** algebra with $\dim \mathcal{A}_\varphi = \mathrm{rank}\, H_\varphi < \infty$.

Kronecker theorems

Univariate series:

Kronecker (1881)

The Hankel operator

$$\begin{array}{rcl} H_\varphi : \mathbb{C}^{\mathbb{N},finite} & \to & \mathbb{C}^{\mathbb{N}} \\ (p_m) & \mapsto & (\sum_m \varphi_{m+n} p_m)_{n\in\mathbb{N}} \end{array}$$

is of **finite rank** r iff $\exists \omega_1, \ldots, \omega_{r'} \in \mathbb{C}[z]$ and $\xi_1, \ldots, \xi_{r'} \in \mathbb{C}$ distincts s.t.

$$\varphi(z) = \sum_{n\in\mathbb{N}} \varphi_n \frac{z^n}{n!} = \sum_{i=1}^{r'} \omega_i(z) \mathbf{e}_{\xi_i}(z)$$

with $\sum_{i=1}^{r'} (\deg(\omega_i) + 1) = r$.

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Multivariate series:

Theorem: Generalized Kronecker Theorem [M'2018]

For $\varphi \in \mathcal{R}^*$, the Hankel operator

$$\begin{array}{rcl} H_\varphi : \mathcal{R} & \to & \mathcal{R}^* \\ p & \mapsto & p \star \varphi \end{array}$$

is of **rank r** iff

$$\varphi(\boldsymbol{z}) = \sum_{\mathrm{i}=1}^{\mathrm{r}'} \omega_{\mathrm{i}}(\boldsymbol{z})\, \boldsymbol{e}_{\xi_{\mathrm{i}}}(\boldsymbol{z}) \quad \text{with } \omega_i(\boldsymbol{z}) \in \mathbb{K}[\boldsymbol{z}],$$

with $r = \sum_{i=1}^{r'} \dim\langle\langle \omega_i(\boldsymbol{z})\rangle\rangle = \sum_{i=1}^{r'} \dim\langle \partial_{\boldsymbol{z}}^{\gamma} \omega_i(\boldsymbol{z})\rangle$. In this case, we have

- $I_\varphi = \ker H_\varphi$ with $\mathcal{V}_{\mathbb{C}}(I_\varphi) = \{\xi_1, \ldots, \xi_{r'}\}$.
- $I_\varphi = Q_1 \cap \cdots \cap Q_{r'}$ with $Q_i^\perp = \langle\langle \omega_i \rangle\rangle\, \boldsymbol{e}_{\xi_i}(\boldsymbol{z})$.

☞ $\mathcal{A}_\varphi$ is **Gorenstein**[a]; $(a, b) \mapsto \langle \varphi | ab \rangle$ is non-degenerate in $\mathcal{A}_\varphi$.

☞ Can be generalized to $\varphi = (\varphi_1, \ldots, \varphi_m) \in (\mathcal{R}^*)^m$.

[a] $\mathcal{A}_\varphi^* = \mathcal{A}_\varphi \star \varphi$ is a free $\mathcal{A}_\varphi$-module of rank 1

For $F \in \mathcal{S}^d$,

- $\mathrm{rank}_{gad}(F)$ minimal $r = \sum_i \dim\langle\langle \omega_i^{d,\ell_i,\boldsymbol{x}} \rangle\rangle$ with $F = \sum_i \omega_i \ell_i^{d-k_i}$.
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Theorem: **[Barrilli-M-Taufer'2025]**

Let $F = \sum_i \omega_i \ell_i^{d-k_i} \in \mathcal{S}^d$ such that $\omega_i \in \mathcal{S}^{k_i}$ and $\ell_i \in \mathcal{S}^1$.
If the **Castelnuovo-Mumford regularity** of $\mathcal{S}/\underline{I}$ is less than $\frac{d+1}{2}$, where $\underline{I}$ is the ideal associated with the GAD, then

$$\mathrm{rank}_{gad}(f) = \mathrm{rank}_{cactus}(f) = \mathrm{rank}\, H,$$

where H is the Catalecticant matrix of F in degree $(d-c, c)$ with $c = \lfloor \frac{d-1}{2} \rfloor$.

In this case, the decomposition is the **unique minimal GAD**, and defines the **unique minimal apolar scheme** to F.

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☞ Applies for Waring decompositions, tangential decompositions, ...

GAD algorithm [BMT'25]

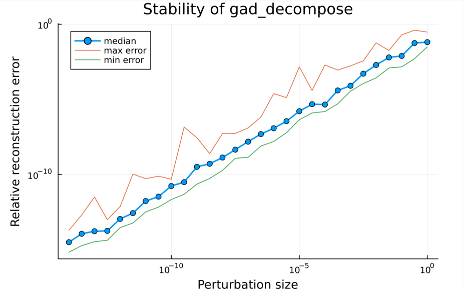
Input: $F \in \mathcal{S}^d$

- Perform a random change of coordinates to obtain $\underline{\xi}_{i,0} \neq 0$
- Compute a basis B, B' via SVD of $H_F^{d-c,c}$
- Build the **multiplication** matrices $M_j = (H_F^{B',B})^{-1} H_F^{B',x_j B}$
- Compute the **multiplicities** μ_i via a **Schur factorisation** of random $\sum_j \lambda_j M_j$.
- Compute the **local multiplication** blocks $M_{x_j}^{(i)}$.
- **Extract the points** $\xi_{i,j} = \frac{1}{\mu_i}\mathrm{trace}(M_{x_j}^{(i)})$ and set $\ell_i = x_0 + x_1\underline{\xi}_{i,1} + \cdots + x_n\underline{\xi}_{i,n}$.
- Compute the **nil-indices** $\nu_i = k_i + 1$ of $M_{x_j}^{(i)} - \underline{\xi}_{i,j}\mathrm{Id}$.
- if $k_i > d$ then **error:** a nilpotency index exceeds the degree bound.
- Compute $\omega_i \in \mathcal{S}_{k_i}$ by **solving the Vandermonde-like system**

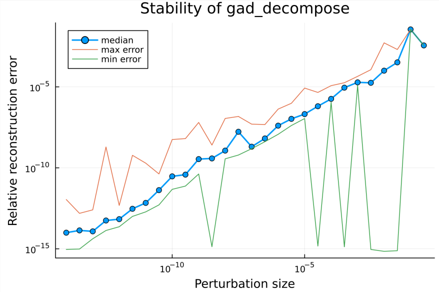
$f = \sum_i \omega_i \ell_i^{d-k_i}$, using the least square method.

Output: ℓ_i, ω_i, μ_i such that $f = \sum_i \omega_i \ell_i^{d-k_i}$ and $\mathrm{rank}_{gad}(f) = \sum_i \mu_i$.

Experimentation for pertubations $F + \epsilon R$ with $F = \sum_{i=1}^{r} \omega_i \ell_i^{d-k_i}$, ℓ_i random in $\mathcal{S}^1$, ω_i random in $\mathcal{S}^{k_i}$ and R random in $\mathcal{S}^d$ (normal centered distributions), corresponding to the format $(n, d, [k_1, k_2, \ldots])$



(a) Case $(n, d, \mathrm{k}) = (9, 3, [0, 0, 0, 0, 0])$: Waring decomposition - Five simple points with $\mathrm{rank}_{gad}(F) = 5$.



(b) Case $(n, d, \mathrm{k}) = (2, 5, [1, 1, 0])$: Two points of multiplicity 2 and one simple point with $\mathrm{rank}_{gad}(F) = 5$.

Joint work with E. Barrilli and D. Taufer.

Tensor extensions

When $\mathrm{reg}(\mathcal{S}/\underline{I})$ is higher, we use the **moment extension** or **extensor** approach [BCMT'10].

Extensor problem:

☞ Find $\tilde{F} \in \mathcal{S}^{d'}$, $d' \geq d$ s.t. $\check{\tilde{F}}^{[\leq d]} = \check{F}$ and $\mathrm{rank}_{\bullet}(\tilde{F}) = \mathrm{rank}_{\bullet}(F)$.

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Flat extension of (Hankel) matrices:

$$\mathsf{H}^{\mathsf{C},\mathsf{C}'} = \left[\begin{array}{c|c} \mathsf{H}^{\mathsf{B},\mathsf{B}'} & H^{B,\overline{B}'} \\ \hline H^{\overline{B},B'} & H^{\overline{B},\overline{B}'} \end{array}\right]$$

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☞ Find $\tilde{F} \in \mathcal{S}^{d'}$, $d' \geq d$ s.t. $\check{\tilde{F}}^{[\leq d]} = \check{F}$ and $\mathrm{rank}_\bullet(\tilde{F}) = \mathrm{rank}_\bullet(F)$.

Flat extension of (Hankel) matrices:

$$\mathsf{H}^{\mathsf{C},\mathsf{C}'} = \left[\begin{array}{c|c} \mathsf{H}^{\mathsf{B},\mathsf{B}'} & H^{B,\overline{B}'} \\ \hline H^{\overline{B},B'} & H^{\overline{B},\overline{B}'} \end{array}\right]$$

Flat extension when

$$\mathrm{rank}\,\mathsf{H}^{\mathsf{C},\mathsf{C}'} = \mathrm{rank}\,\mathsf{H}^{\mathsf{B},\mathsf{B}'}$$

Let $R = \mathbb{K}[\boldsymbol{x}]$ and "set" $x_0 = 1$.

Definition: For $B \subset R$,

- $B^+ = B \cup x_1 \cdot B \cup \cdots \cup x_n \cdot B$,
- B **connected to 1** if $1 \in B$ and for $m \neq 1 \in B$, $\exists m' \in B, i_m \in [n] : m = x_{i_m} m'$.

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Theorem: **[Laurent-M'09]**

Let $B \subset C \subset R, B' \subset C' \subset R$ connected to 1 and $B^+ \subset C, B'^+ \subset C'$. Assume $H^{B',B}$ invertible with $|B| = |B'| = r$. The following points are equivalent:

- $H^{C',C}$ is a **flat extension** of $H^{B',B}$.
- The operators $M_j := (H^{B',B})^{-1} H^{B',x_j B}$ **commute**.

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Theorem: **[Brachat-Comon-M-Tsigaridas'10]**

$F \in \mathcal{S}^d$ is of rank r iff $\exists\, \tilde{F} \in S^{d'}$ of F with $\check{\tilde{F}}^{[\leq d]} = \check{F}$ and $d' \geq 2r+1$ s.t.

- $\mathrm{rank} H_{\tilde{F}}^{d'-c,c} = r$,
- $H_{\tilde{F}}^{d'-c,c}$ is a **flat extension** of $H_{\tilde{F}}^{B,B'}$ with B'$^+ \subset R_{\leq c}$.

☞ For the Waring rank, $M_j = (H_{\tilde{F}}^{B',B})^{-1} H_{\tilde{F}}^{B',x_j B}$ are **jointly diagonalizable**.

Joint diagonalization (Joint work with Chuong Luong)

Problem

i) Compute the moments $\varphi_\alpha = \int x^\alpha d\mu$ for a measure μ for $|\alpha| \le d$.

ii) Decompose
$F(\mathbf{x}) = \int (1 + (\mathbf{x}, \mathbf{y}))^d d\mu(\mathbf{y}) = \sum_{|\alpha| \le d} \varphi_\alpha \binom{d}{\alpha} \mathbf{x}^\alpha = \sum_{i=1}^r \omega_i \left(1 + (\xi_i, \mathbf{x})\right)^d$

Joint Diagonalization

1. using the single diagonalisation of a random combination of the M_i, or
2. by minimization of $\min_{E \text{ inv.}} \sum_i \|E M_i E^{-1}\|_{\text{off}}$ with Jacobi updates $E_{k+1} = (I + X_k)\, E_k$ and gradient descent. [P. Catalat]

Computing the weights

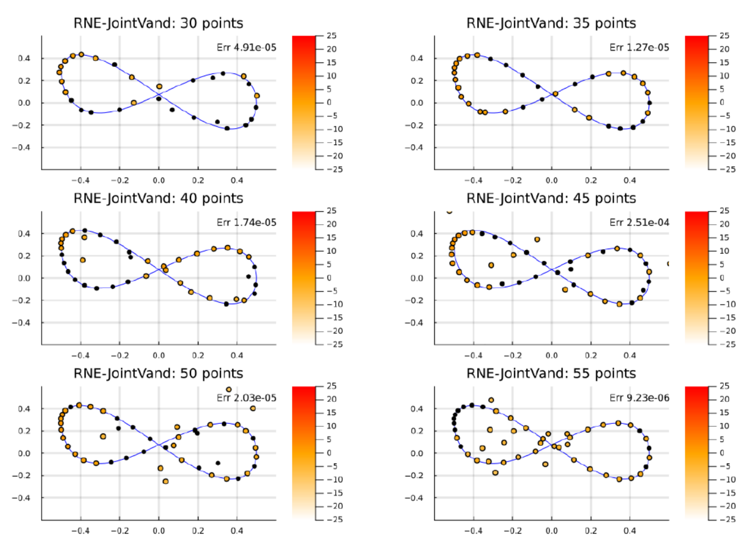
a) explicit formulae from the joint eigenvectors, or

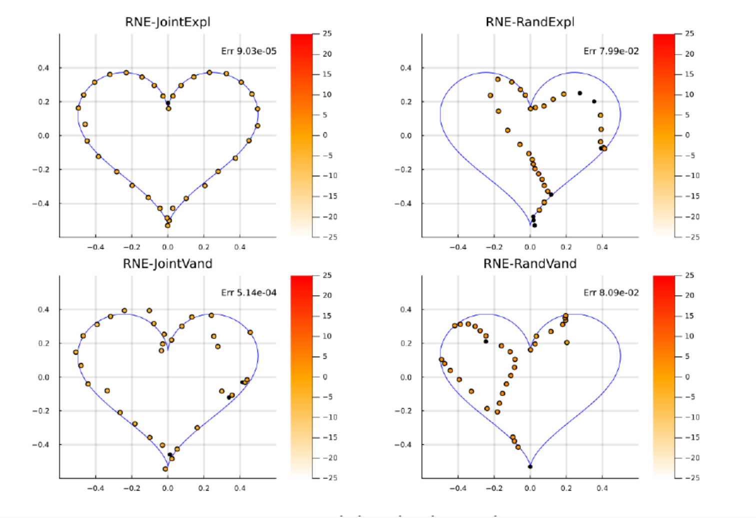
b) solving a Vandermonde system $V_{A,\Xi}\, \omega = B$.

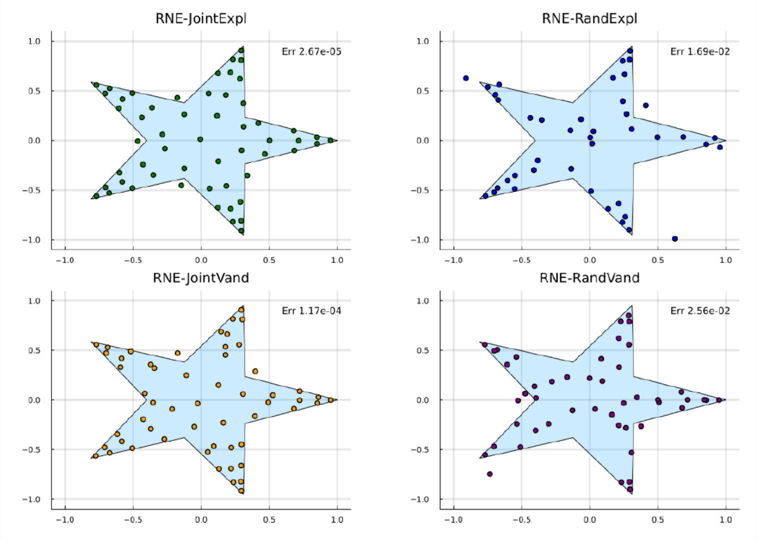
Improving the decomposition

- Minimization of $\|F - \sum_i \omega_i (\xi, \mathbf{x})^d\|$ with Riemannian Newton steps (RNE) and trust-region scheme [R. Khouja]

Implemented in `TensorDec.jl`


RNE-JointVand: 30 points
Err 4.91e-05
RNE-JointVand: 35 points
Err 1.27e-05
RNE-JointVand: 40 points
Err 1.74e-05
RNE-JointVand: 45 points
Err 2.51e-04
RNE-JointVand: 50 points
Err 2.03e-05
RNE-JointVand: 55 points
Err 9.23e-06


RNE-JointExpl
Err 9.03e-05
RNE-RandExpl
Err 7.99e-02
RNE-JointVand
Err 5.14e-04
RNE-RandVand
Err 8.09e-02


RNE-JointExpl
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RNE-RandExpl
Err 1.69e-02
RNE-JointVand
Err 1.17e-04
RNE-RandVand
Err 2.56e-02
−1.0
−0.5
0.0
0.5
1.0

Thanks for your attention

Acknowlegments:

TENORS DN-JD MCSA (tenors-network.eu)