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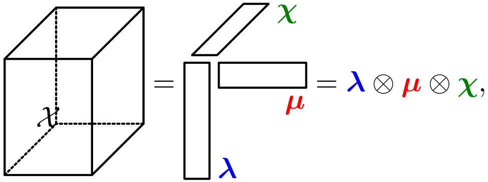
# Introduction to tensor rank decomposition

# I. The tensor rank decomposition for data analysis

Pure fluorophores dissolved in water

The **intensity** $x_{i,j,k}$ of the light that is emitted at wavelength $\omega_j$ when a fluorophore, diluted in water with a concentration $c_k$, is excited at wavelength $\omega_i$ forms a **rank-1 tensor**:[1]

$$x_{i,j,k} = \lambda_i \cdot \mu_j \cdot \chi_k$$



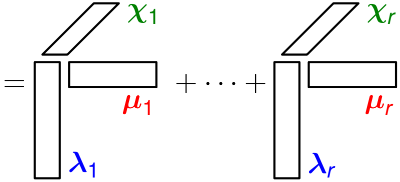
where

- $\lambda_i$ is the fraction of light absorbed at wavelength $\omega_i$,
- $\mu_j$ is the fraction of light emitted at wavelength $\omega_j$, and
- $\chi_k$ is a constant proportional to the concentration $c_k$.

[1] Appellof & Davidson, Anal. Chem., 1981.

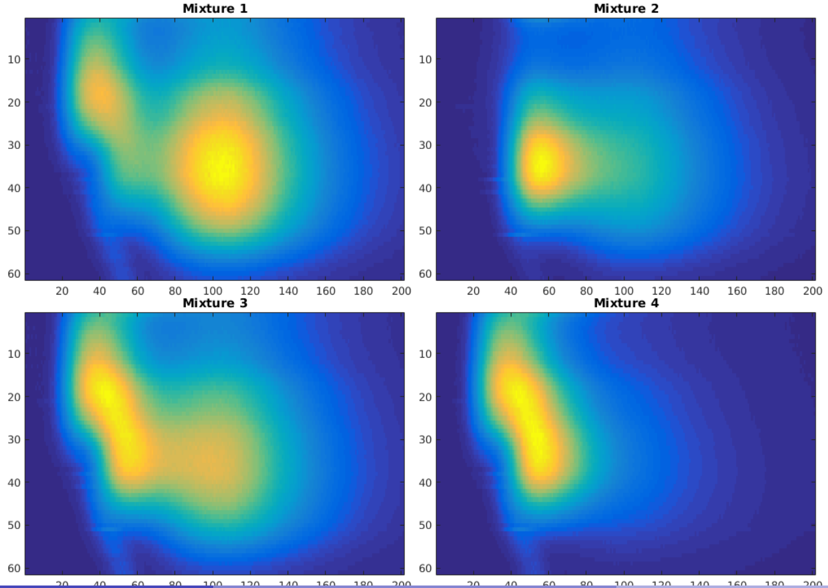
If several mixtures are simultaneously analyzed, then the intensity $x_{i,j,k}$ of the emitted light at the *j*th output wavelength when the *k*th mixture is excited at the *i*th input wavelength forms a **tensor rank decomposition**:

$$\mathcal{X} = \sum_{\ell=1}^{r} \boldsymbol{\lambda}_\ell \otimes \boldsymbol{\mu}_\ell \otimes \boldsymbol{\chi}_\ell$$

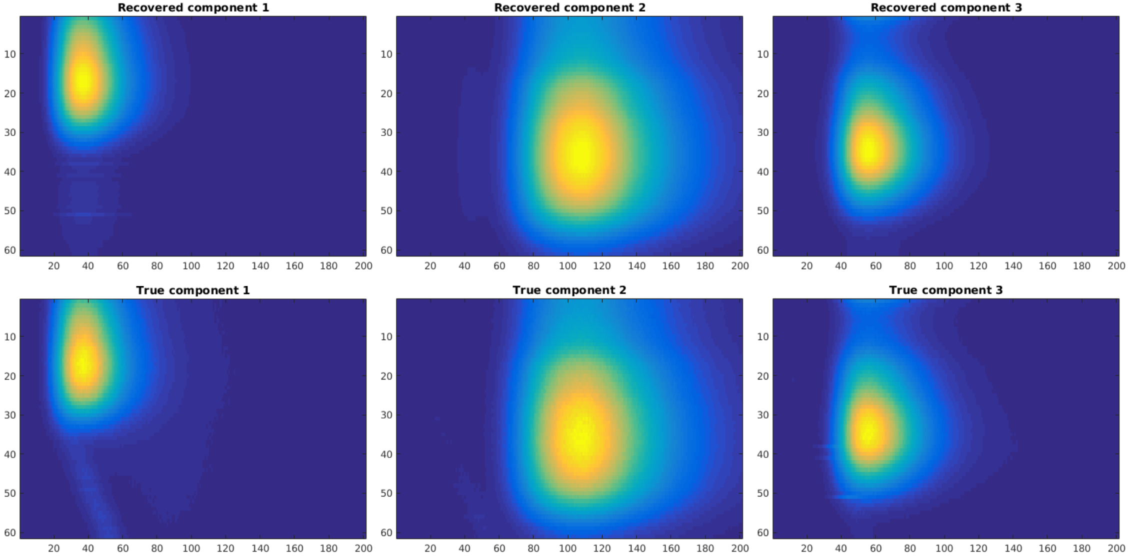



Typically, the fluorophores can be recovered uniquely!

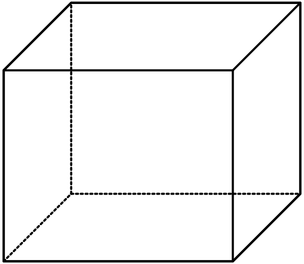
We made 4 random mixtures of the three pure fluorophores:



Then, we recovered the spectra from a rank-3 decomposition:



# II. Tensors



# Multilinear maps

## Definition

Let $U_k$ for $k = 1, \ldots, d$ and $W$ be vector spaces over a field $\mathbb{F}$ of characteristic 0.
A **multilinear map** is a map $\phi : U_1 \times \cdots \times U_d \to W$ which is linear in each of its arguments individually:

$$\phi(\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}, (\alpha\mathbf{x} + \beta\mathbf{y}), \mathbf{u}_{k+1}, \ldots, \mathbf{u}_d) = \\ \alpha \cdot \phi(\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}, \mathbf{x}, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_d) + \beta \cdot \phi(\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}, \mathbf{y}, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_d)$$

for every $k = 1, \ldots, d$, all $\alpha, \beta \in \mathbb{F}$, and all $\mathbf{x}, \mathbf{y} \in U_k$.

If $d = 1$, we have a regular **linear map**.

[1] Greub, Multilinear Algebra, 1978.

Some examples of multilinear maps include:

1. complex multiplication $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$;
2. the standard inner product $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$;
3. matrix chain multiplication $(M_1, \ldots, M_d) \mapsto M_1 \cdots M_d$;
4. the determinant $\mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$;
5. polynomial multiplication of linear forms $\mathbb{R}[\mathbf{x}]_{\leq 1} \times \cdots \times \mathbb{R}[\mathbf{x}]_{\leq 1} \to \mathbb{R}[\mathbf{x}]_{\leq d}$;

# Tensor product

## Definition

Let $U_1, U_2, \dots, U_d$ be vector spaces over a field $\mathbb{F}$. The **tensor product** of these vector spaces is a vector space $U_1 \otimes U_2 \otimes \cdots \otimes U_d$, together with a multilinear map

$$\otimes : U_1 \times U_2 \times \cdots \times U_d \longrightarrow U_1 \otimes U_2 \otimes \cdots \otimes U_d$$

that satisfies the following **universal property**: For every vector space $W$ and every multilinear map

$$\phi : U_1 \times U_2 \times \cdots \times U_d \longrightarrow W,$$

there exists a unique linear map

$$f : U_1 \otimes U_2 \otimes \cdots \otimes U_d \longrightarrow W$$

such that $\phi = f \circ \otimes$.

[1] Greub, Multilinear Algebra, 1978.

The universal property of $\otimes$ says for every multilinear $\phi : U_1 \times \cdots \times U_d \to W$ there is a unique linear map $f$ such that we have the commuting diagram

$$
\begin{tikzcd}
U_1 \times \cdots \times U_d \arrow[r, "\phi"] \arrow[d, "\otimes"'] & W \\
U_1 \otimes \cdots \otimes U_d \arrow[ur, "f"'] &
\end{tikzcd}
$$

That is, $\otimes$ is The One Multilinear Map to rule them all.

[1] Image by Peter J. Yost.

By invoking the universal property twice, we can show that two tensor products $\otimes_1$ and $\otimes_2$ on the same tuple of vector spaces yield **uniquely isomorphic tensor product spaces.** Indeed:

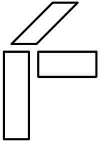
$$\begin{tikzcd}
U_1 \times \cdots \times U_d \arrow[r, "\otimes_2"] \arrow[d, "\otimes_1"'] & U_1 \otimes_2 \cdots \otimes_2 U_d \arrow[dl, shift right, "f"'] \\
U_1 \otimes_1 \cdots \otimes_1 U_d \arrow[ur, shift right, "g"'] &
\end{tikzcd}$$

In this sense, the tensor product on a tuple of vector spaces is **essentially unique.**

It can be shown the **Segre map**

$$\mathrm{Seg} : \mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \times \cdots \times \mathbb{F}^{n_d} \to \mathbb{F}^{n_1 \times \cdots \times n_d}$$

$$(\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z}) \mapsto \mathbf{x} \otimes \mathbf{y} \otimes \cdots \otimes \mathbf{z} := [x_{i_1} y_{i_2} \cdots z_{i_d}]_{i_1,\ldots,i_d} =$$



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is the **tensor product** for these vector spaces.

# Tensors in coordinates: multidimensional arrays

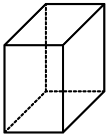
Uniqueness of the linear map implies that $U_1 \otimes \cdots \otimes U_d = \mathrm{span}(\mathrm{im}(\otimes))$.
If $\{\mathbf{u}_{k1}, \ldots, \mathbf{u}_{kn_k}\}$ is a basis of $U_k$, then

$$\mathcal{B} = \{\mathbf{u}_{1i_1} \otimes \cdots \otimes \mathbf{u}_{di_d} \mid i_k = 1, \ldots, n_k, \;\; k = 1, \ldots, d\}$$

is a **tensor product basis** of $U_1 \otimes \cdots \otimes U_d$.

The coefficients of $\mathcal{A} \in U_1 \otimes \cdots \otimes U_d$ with respect to a tensor product basis $\mathcal{B}$ are then naturally organized as the $d$-array $\mathcal{A} \in \mathbb{F}^{n_1 \times \cdots \times n_d}$:

$$\mathcal{A} = \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} a_{i_1,\ldots,i_d} \mathbf{u}_{1i_1} \otimes \cdots \otimes \mathbf{u}_{di_d} =$$



# Tensor product of linear maps

Since the space of linear maps from $U_i$ to $V_i$ is a vector space, $\mathrm{Hom}(U_i, V_i)$, we can take the **tensor product of linear maps**:[2]

$$\mathrm{Hom}(U_1, V_1) \otimes \cdots \otimes \mathrm{Hom}(U_d, V_d) \simeq \mathrm{Hom}(U_1 \otimes \cdots \otimes U_d, V_1 \otimes \cdots \otimes V_d).$$

If $A_i \in \mathrm{Hom}(U_i, V_i)$, then the linear map $A_1 \otimes \cdots \otimes A_d$ is uniquely defined by its action on elementary tensors due to universality:

$$(A_1 \otimes \cdots \otimes A_d)(\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_d) := (A_1 \mathbf{u}_1) \otimes \cdots \otimes (A_d \mathbf{u}_d).$$

Applying a tensor product of linear maps is called a **multilinear multiplication** in numerical multilinear algebra.

[2] Greub, Multilinear Algebra, 1978.

# Tensor decompositions

A **tensor decomposition** of a tensor is an expression that picks up on some special structure of $\mathcal{A} \in V_1 \otimes \cdots \otimes V_d$.

Two natural structures come to mind:

1. A tensor $\mathcal{A}$ could live in a **tensor product subspace** $U_1 \otimes \cdots \otimes U_d \subseteq V_1 \otimes \cdots \otimes V_d$.
2. Every tensor $\mathcal{A} \in V_1 \otimes \cdots \otimes V_d$ can be expressed as

$$\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_r,$$

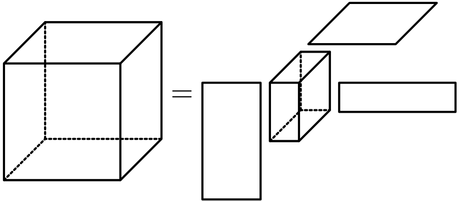
with each $\mathcal{A}_i \in \mathrm{im}(\otimes)$, because $V_1 \otimes \cdots \otimes V_d = \mathrm{span}(\mathrm{im}(\otimes))$.

Given $\mathcal{A} \in V_1 \otimes \cdots \otimes V_d$, there is a **unique tensor product subspace**

$$U_1 \otimes \cdots \otimes U_d \subseteq V_1 \otimes \cdots \otimes V_d$$

of **minimal dimension** containing $\mathcal{A}$. If $\imath_i : U_i \hookrightarrow V_i$ is the canonical inclusion map, then

$$\mathcal{A} = (\imath_1 \otimes \cdots \otimes \imath_d)(\mathcal{A})$$

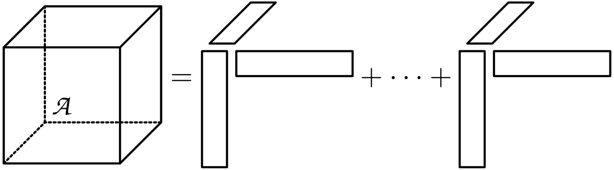


is called a **minimal Tucker decomposition** of $\mathcal{A}$.[3]

[3]Tucker, Psychometrica, 1966; but also essentially Hitchcock, J. Math. Phys., 1927.

Given $\mathcal{A} \in V_1 \otimes \cdots \otimes V_d$, there are expressions of the form

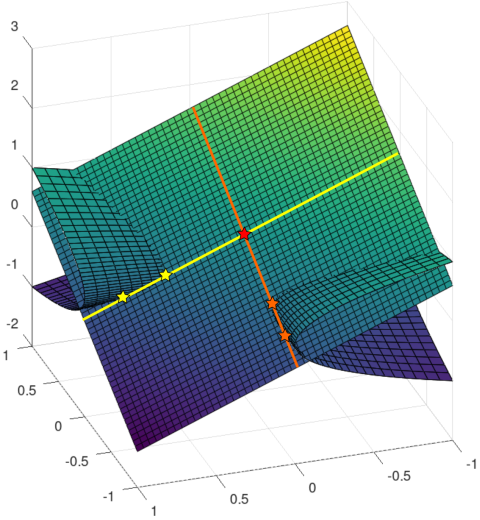
$$\mathcal{A} = \sum_{i=1}^{r} \mathbf{a}_i^1 \otimes \cdots \otimes \mathbf{a}_i^d$$




The ones of minimum length are called **tensor rank decompositions** of $\mathcal{A}$. This decomposition was introduced and studied by Hitchcock.[4]

[4] Hitchcock, J. Math. Phys., 1927.

# III. Geometry of tensor rank decomposition



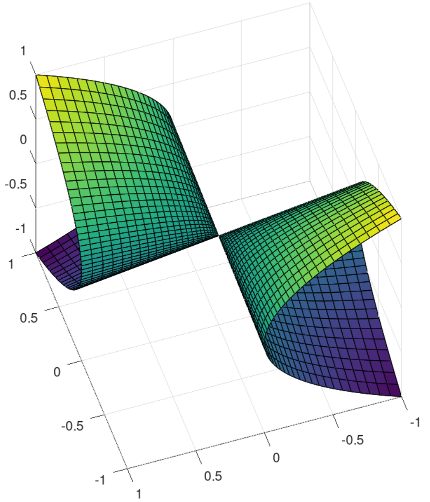
# The Segre variety

The Segre map is a **polynomial map.**

Hence, the image of $\mathrm{Seg} = \otimes$ is a **constructible set** or **quasi-algebraic variety.**[a] One can prove $\mathcal{S} = \mathrm{im}(\mathrm{Seg})$ is closed, so

$$\mathcal{S} = \{\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_d \mid \mathbf{u}_k \in \mathbb{F}^{n_k}, \quad k = 1, \ldots, d\}$$

is an **irreducible algebraic variety.**




$\mathcal{S} \subset S^2\mathbb{F}^2$

[a] Cox, Little, & O'Shea, Ideals, Varieties, and Algorithms, 2025.

The Segre variety is well-studied:[5]

1. The **ideal-theoretic equations** are known; they are determinantal.
2. The **tangent space** at $p = \mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_d \in \mathcal{S}$ is

$$\mathrm{T}_p\mathcal{S} = \mathbb{R}^{n_1} \otimes \mathbf{u}_2 \otimes \cdots \otimes \mathbf{u}_d + \cdots + \mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_{d-1} \otimes \mathbb{R}^{n_d}.$$

3. Its **dimension** is

$$\dim \mathcal{S} = 1 + (n_1 - 1) + \cdots + (n_d - 1).$$

4. The only singular point of $\mathcal{S}$ is 0; $\mathcal{S} \setminus \{0\}$ is a **smooth embedded submanifold** of $\mathbb{F}^{n_1 \times \cdots \times n_d}$.
5. It is a **toric variety**.
6. In projective space, $\mathcal{S} = \{\langle \mathcal{A} \rangle \mid \mathcal{A} \in \mathcal{S} \setminus \{0\}\}$ is a **projective variety**.
7. Its **geodesics** are known in the ambient metric and some warped metrics.[6]

[5] Harris, Algebraic Geometry: A First Course, 1992.

[6] Swijsen, Van der Veken, Vannieuwenhoven, Numer. Linear Algebra Appl., 2022.; Jacobsson, Swijsen, Van der Veken, & Vannieuwenhoven, arXiv, 2024.

# Tensor rank/join decompositions

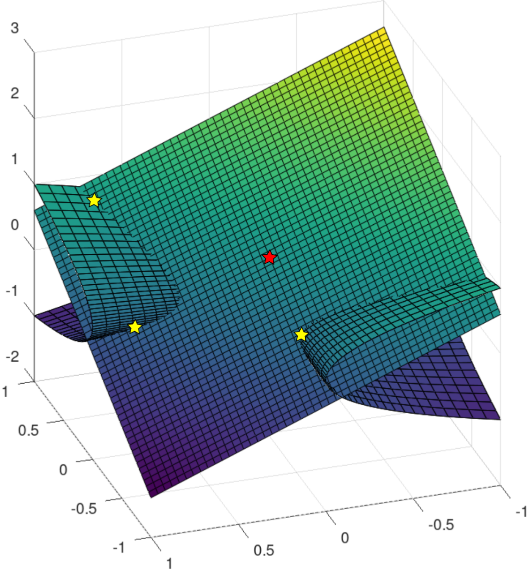
The tensor rank decomposition, a minimum-length expression

$$\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_r$$

is one example of a family of **rank or join decompositions**, where all $\mathcal{A}_i \in \mathcal{V}_i$, for varieties $\mathcal{V}_i \subset V$ in a common ambient vector space $V$.

| **Variety $\mathcal{V}_i$** | **Decomposition** |
|---|---|
| Veronese | Waring (i.e. powers of linear forms $\ell_i^d$) |
| Segre | Tensor rank, canonical polyadic, CPD, CANDECOMP, PARAFAC |
| Chow | Chow (i.e. product of linear forms $\ell_i \cdots m_i$) |
| Segre–Veronese | Partially symmetric |
| Chow–Veronese | Chow–Waring |
| Subspace | Block term (i.e. tensor product subspaces $(\imath_1 \otimes \cdots \otimes \imath_d)(\mathcal{A}_i)$) |
| Grassmannian | Grassmann (i.e. subspaces $\mathbf{u}_i \wedge \mathbf{v}_i \wedge \cdots \wedge \mathbf{w}_i$) |

Decompositions $\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_r, \quad \mathcal{A}_i \in \mathcal{V}$ have a **nice geometric interpretation**:



# Secant varieties

Let $\mathcal{V} \subset V$ be an algebraic variety that is a **cone** ($\mathcal{X} \in \mathcal{V}$ if and only if $\lambda\mathcal{X} \in \mathcal{V}$ for $\lambda \in \mathbb{F}_*$). Then,

$$\begin{aligned}\Sigma_r : \mathcal{V} \times \cdots \times \mathcal{V} &\longrightarrow V,\\ (\mathcal{A}_1, \ldots, \mathcal{A}_r) &\longmapsto \mathcal{A}_1 + \cdots + \mathcal{A}_r\end{aligned}$$

is a rational morphism of varieties. Hence, its image will be a constructible set.

The **Zariski closure** of $\mathrm{im}(\Sigma_r)$ is called the ***r*th secant variety of** $\mathcal{V}$:[7]

$$\sigma_r(\mathcal{V}) = \overline{\mathrm{im}(\Sigma_r)}.$$

Note that $\Sigma_r$ is a dominant morphism onto $\sigma_r(\mathcal{V})$.

[7] Landsberg, Tensors: Geometry and Applications, 2012.
Bernardi, Carlini, Catalisano, Gimigliano, & Oneto, Mathematics, 2018.

The $r$th secant variety $\sigma_r(\mathcal{V})$ consists of all vectors in $V$ that can be expressed as a sum of at most $r$ points of $\mathcal{V}$. Naturally, we have an **ascending chain**

$$\mathcal{V} = \sigma_1(\mathcal{V}) \subsetneq \sigma_2(\mathcal{V}) \subsetneq \cdots \subsetneq \sigma_R(\mathcal{V}) = \sigma_{R+1}(\mathcal{V}) = \dots .$$

If the chain stabilizes at $V$, then $\mathcal{V}$ is called **nondegenerate**, i.e., if $V = \mathrm{span}(\mathcal{V})$.

The unique smallest value $R$ for which $\sigma_R(\mathcal{V}) = V$ is called the **generic $\mathcal{V}$-rank.**

# Nondefectivity

Since

$$\mathrm{d}\Sigma_r : \mathrm{T}\mathcal{V} \times \cdots \times \mathrm{T}\mathcal{V} \to V, \quad (\dot{\mathcal{A}}_1, \ldots, \dot{\mathcal{A}}_r) \longmapsto \dot{\mathcal{A}}_1 + \cdots + \dot{\mathcal{A}}_r,$$

we have the following result.[8]

**Lemma (Terracini, 1911)**

*The tangent space of $\sigma_r(\mathcal{V})$ at $\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_r$ satisfies*

$$\mathrm{T}_{\mathcal{A}_1}\mathcal{V} + \cdots + \mathrm{T}_{\mathcal{A}_r}\mathcal{V} \subseteq \mathrm{T}_{\mathcal{A}}\sigma_r(\mathcal{V}).$$

*On a Zariski-open subset of $\sigma_r(\mathcal{S})$ equality holds.*

The matrix of $\mathrm{d}\Sigma_r$ w.r.t. some bases is called **Terracini's matrix.**

[8] Terracini, Rend. Circ. Mat. Palermo, 1911.

As a corollary of Terracini's lemma, we have

$$\dim \sigma_r(\mathcal{V}) = \dim\left(\mathrm{T}_{\mathcal{A}_1}\mathcal{V} + \cdots + \mathrm{T}_{\mathcal{A}_r}\mathcal{V}\right) \le \min\{r \cdot \dim \mathcal{V}, \dim V\} =: \dim_E \sigma_r(\mathcal{V})$$

for generic $\mathcal{A}_i \in \mathcal{V}$.

If equality holds, i.e., the **expected dimension** $\dim_E \sigma_r(\mathcal{V})$ coincides with the actual dimension $\dim \sigma_r(\mathcal{V})$, then $\mathcal{V}$ is called ***r*-nondefective.** Otherwise it is called ***r*-defective.**

If $r$-nondefectivity holds for all $r$, then $\mathcal{V}$ is **nondefective.** The generic rank satisfies

$$\mathrm{grank}(\mathcal{V}) \ge \left\lceil \frac{\dim V}{\dim \mathcal{V}} \right\rceil,$$

with equality for a nondefective $\mathcal{V}$.

It is a difficult question to determine the dimension of secant varieties of arbitrary varieties. To my knowledge, the state of the art is as follows:

| Variety | Status | Reference |
|---|---|---|
| Veronese | Resolved | Alexander & Hirschowitz (1995) |
| Segre | Mostly resolved | Abo, Ottaviani, & Peterson (2008); Chiantini, Ottaviani, & Vannieuwenhoven (2014); Blomenhofer & Casarotti (2024) |
| Chow | Mostly resolved | Torrance & Vannieuwenhoven (2021); Blomenhofer & Casarotti (2024) |
| Segre–Veronese | Much progress | Abo, Bambilla, Galuppi, Oneto (2024); Ballico (2024) |
| Chow–Veronese | Mostly open | Bernardi, Catalisano, Gimigliano, & Idà (2009); Abo & Vannieuwenhoven (2018) |
| Subspace | Open | Yang (2014) |
| Grassmannian | Mostly resolved | Blomenhofer & Casarotti (2024) |

# Identifiability

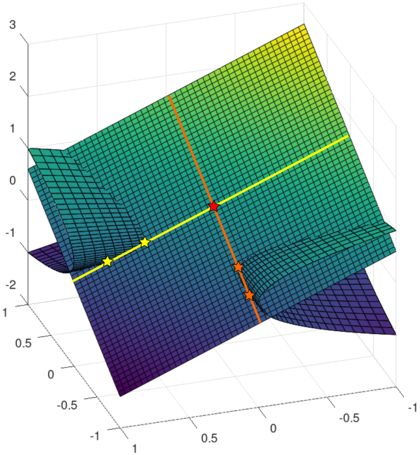
The generic fiber of

$$\Sigma_r : \mathcal{V} \times \cdots \times \mathcal{V} \to \sigma_r(\mathcal{V}),$$
$$(\mathcal{A}_1, \ldots, \mathcal{A}_r) \mapsto \mathcal{A}_1 + \cdots + \mathcal{A}_r$$

is **finite** only if the $r$th secant variety is nondefective. That is, a generic $\mathcal{A} \in \sigma_r(\mathcal{V})$ has a finite number of decompositions.

For a nondefective $r$th secant variety, the unique cardinality of the generic fiber of $\Sigma_r$ is called the $r$**-degree.**

If the degree of $\Sigma_r$ is one, $\mathcal{V}$ is called $r$**-identifiable.**



Varieties $\mathcal{V}$ that are $r$-nondefective but not $r$-identifiable are special! Suppose that a generic $\mathcal{A} \in \sigma_r(\mathcal{V})$ has at least two distinct decompositions:

$$\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_r = \mathcal{A}_1' + \cdots + \mathcal{A}_r'.$$

By Terracini's lemma, we will have

$$\mathrm{T}_{\mathcal{A}_1}\mathcal{V} + \cdots + \mathrm{T}_{\mathcal{A}_r}\mathcal{V} = \mathrm{T}_{\mathcal{A}}\sigma_r(\mathcal{V}) \supseteq \mathrm{T}_{\mathcal{A}_1'}\mathcal{V} + \cdots + \mathrm{T}_{\mathcal{A}_r'}\mathcal{V}.$$

Moving the points $\mathcal{A}_1, \ldots, \mathcal{A}_r$ in the space they span does not change $\mathrm{T}_{\mathcal{A}_1}\mathcal{V} + \cdots + \mathrm{T}_{\mathcal{A}_r}\mathcal{V}$, while the alternative decomposition $\mathcal{A}_1' + \cdots + \mathcal{A}_r'$ must move (otherwise they lie in the same subspace, contradicting the minimality of the rank of $\mathcal{A}$).

The foregoing argument leads to the following concept.[9]

### Definition (Tangential weak defectivity)

A variety $\mathcal{V}$ is ***r*-tangentially weakly defective** if the $r$-tangential contact locus

$$\mathcal{C}_r = \{p \in \mathcal{V} \mid \mathrm{T}_p\mathcal{V} \subset \mathrm{T}_{\mathcal{A}}\mathcal{V}\}$$

has strictly positive dimension for generic $\mathcal{A} \in \sigma_r(\mathcal{V})$.
A variety is **not *r*-twd** if the $r$-tangential contact locus at a generic $\mathcal{A}$ consist of $r$ points.

We have the following chain of implications for a variety $\mathcal{V}$:

$$\text{not } r\text{-weakly defective} \implies \text{not } r\text{-twd} \implies r\text{-identifiable} \implies r\text{-nondefective}$$

and smoothness of $\mathcal{V}$ implies not 1-weak defectivity.

[9] Chiantini & Ottaviani, SIAM J. Matrix Anal. Appl., 2012

Massarenti & Mella (2024), building on Casarotti & Mella (2022), established the following fabulous result.

**Theorem (Massarenti & Mella, 2024)**

*Let $\mathcal{V}$ be an irreducible, nondegenerate variety. If*

- *$\mathcal{V}$ is $(r+1)$-nondefective,*
- *$r+1 < \mathrm{grank}(\mathcal{V})$, and*
- *$\mathcal{V}$ is not $1$-twd,*

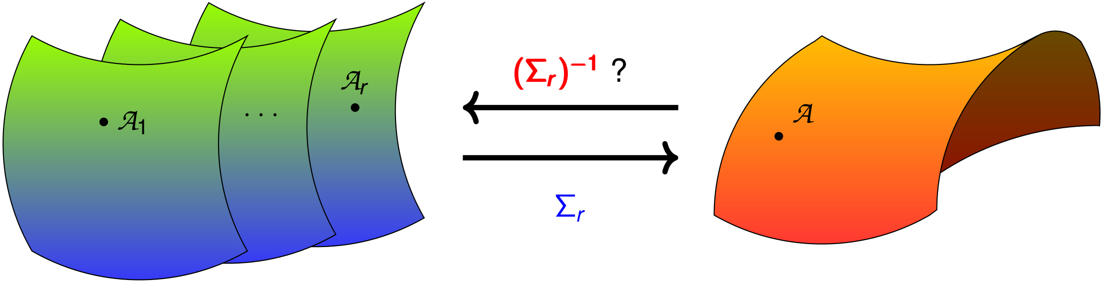
*then $\mathcal{V}$ is $r$-identifiable.*

Note that the following varieties are not 1-twd:

- Veronese, Segre, Segre–Veronese, and Grassmannian (smooth);
- Chow;[10]

[10] Oeding, Adv. Math., 2012.

# IV. Sensitivity of join decompositions



# Condition number

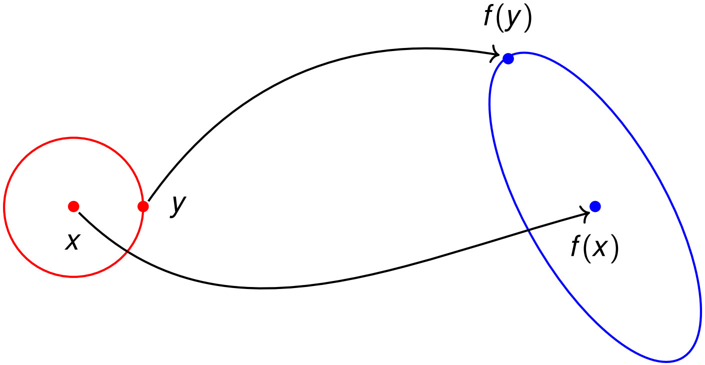
Because of identifiability, the tensor rank decomposition can be employed for **data analysis**, by identifying dominant features or explanatory factors.

However, data tensors arising in applications can be **corrupted** by

- measurement errors,
- roundoff errors,
- computation and simulation errors.

Hence we wish to **quantify** how much these explanatory factors (rank-1 tensors $\mathcal{A}_i$) can vary under **small perturbations** of the data tensor $\mathcal{A}_1 + \cdots + \mathcal{A}_r$.

Rice (1966) defined the **condition number** for maps $f : \mathcal{I} \to \mathcal{O}$ between **metric spaces** $(\mathcal{I}, d_{\mathcal{I}}), (\mathcal{O}, d_{\mathcal{O}})$ as the **worst-case sensitivity** of $f$ to infinitesimal input perturbations.



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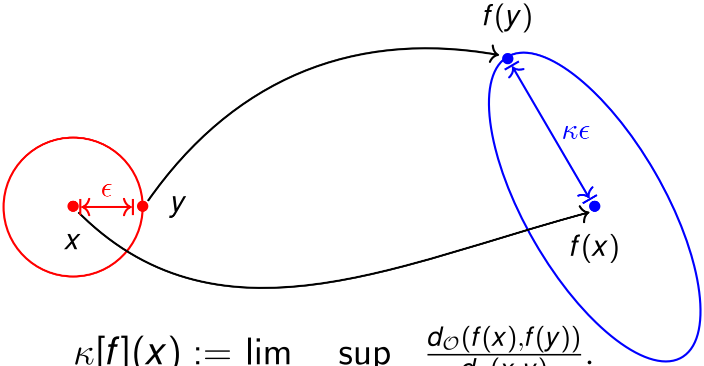


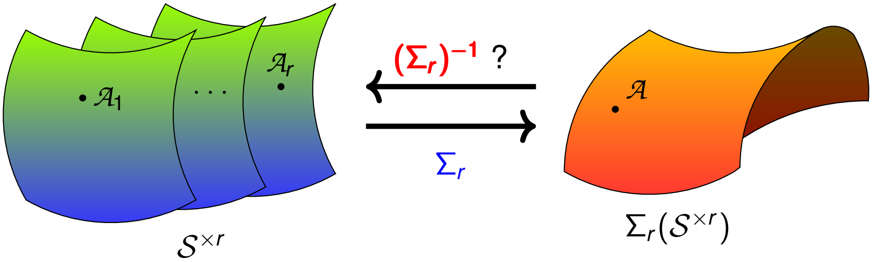

$$\kappa[f](x) := \lim_{\epsilon\to 0} \sup_{\substack{y\in\mathcal{I}, \\ d_\mathcal{I}(x,y)=\epsilon}} \frac{d_\mathcal{O}(f(x),f(y))}{d_\mathcal{I}(x,y)}.$$

# The join decomposition problem

Let $\mathcal{S} \subset V$ be a smooth subvariety of an inner product space $V$. Assume for simplicity $V = \mathbb{R}^N$ is equipped with the standard Euclidean metric.

We want to determine the condition number of a local **inverse** of the **addition map**

$$\Sigma_r : \quad \mathcal{S}^{\times r} \to \mathbb{R}^N$$
$$(\mathcal{A}_1, \ldots, \mathcal{A}_r) \mapsto \mathcal{A}_1 + \cdots + \mathcal{A}_r.$$




This problem was essentially solved by Ulisse Dini in his 1877/1878 *Lezioni d'analisi infinitesimale*.

If $\mathrm{d}_x\Sigma_r$ is injective, the **inverse function theorem** for manifolds states that there exist open neighborhoods

- $\mathcal{O}_x$ of $x = (\mathcal{A}_1, \ldots, \mathcal{A}_r)$ and
- $\mathcal{I}_{\mathcal{A}}$ of $\mathcal{A} = \Sigma_r(x) = \mathcal{A}_1 + \cdots + \mathcal{A}_r$

and a **smooth local inverse function**

$$\Sigma_x^{-1} \circ (\Sigma_r|_{\mathcal{O}_x}) = \mathrm{Id}_{\mathcal{O}_x}.$$

We call $\Sigma_x^{-1} : \mathcal{I}_{\mathcal{A}} \to \mathcal{O}_x$ a **local decomposition map.**

Since this local decomposition map $\Sigma_x^{-1}$ is a **smooth map** we can apply Rice's theorem to Dini's inverse function:

$$\kappa[\Sigma_x^{-1}](\mathcal{A}) = \|\mathrm{d}_{\mathcal{A}}\Sigma_x^{-1}\|_2 = \|(\mathrm{d}_x(\Sigma_r|_{\mathcal{O}_x}))^{-1}\|_2 = \frac{1}{\sigma_{r \dim \mathcal{S}}\,(\mathrm{d}_x(\Sigma_r|_{\mathcal{O}_x}))},$$

where $x = (\mathcal{A}_1, \ldots, \mathcal{A}_r)$, $\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_r$, and $\sigma_i$ is the $i$th largest **singular value.**

Hence, to compute the condition number of $\Sigma_x^{-1}$, it suffices to know the **derivative of** $\Sigma_r$:

$$\begin{aligned} d_x(\Sigma_r|_{\mathcal{O}_x}) : \overbrace{\mathrm{T}_{\mathcal{A}_1}\mathcal{S} \times \cdots \times \mathrm{T}_{\mathcal{A}_r}\mathcal{S}}^{\mathrm{T}_x\mathcal{O}_x} &\to \mathrm{T}_{\mathcal{A}}\mathcal{I}_{\mathcal{A}} \\ (\dot{\mathcal{A}}_1, \ldots, \dot{\mathcal{A}}_r) &\mapsto \dot{\mathcal{A}}_1 + \cdots + \dot{\mathcal{A}}_r. \end{aligned}$$

Representing this derivative in coordinates yields[11] the next

## Definition (Terracini matrix)

The **Terracini matrix** of $\mathcal{A}_1, \ldots, \mathcal{A}_r \in \mathcal{S}$ is

$$T_{\mathcal{A}_1,\ldots,\mathcal{A}_r} = \begin{bmatrix} Q_1 & Q_2 & \cdots & Q_r \end{bmatrix} \in \mathbb{R}^{N \times r \dim \mathcal{S}},$$

where $Q_i \in \mathbb{R}^{N \times \dim \mathcal{S}}$ is a matrix whose columns contain an **orthonormal basis of the tangent space** $\mathrm{T}_{\mathcal{A}_i}\mathcal{S}$.

[11] For more details see Breiding and Vannieuwenhoven (2018).

In conclusion, the following result is obtained.

**Theorem (Breiding and Vannieuwenhoven, 2018)**

*The condition number of computing the join decomposition of* $\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_r$ *at* $x = (\mathcal{A}_1, \ldots, \mathcal{A}_r)$ *with* $\mathcal{A}_i \in \mathcal{S}$ *is*

$$\kappa[\Sigma_x^{-1}](\mathcal{A}) = \frac{1}{\sigma_{r \dim \mathcal{S}}(T_{\mathcal{A}_1,\ldots,\mathcal{A}_r})}.$$

Note that for $r = 1$, the condition number is always 1. (Sensible, because $\Sigma_1 = \mathrm{Id}$.)

# IV. Conclusions

We discussed

- the essentials of multilinear algebra and tensor decompositions,
- the geometry of the Segre variety,
- the geometry of secant varieties,
- nondefectivity, identifiability, tangential weak defectivity, and their relations,
- sensitivity of tensor join decompositions.

# IV. Conclusions

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Merci pour votre attention!

# References


- Abo, Brambilla, Galuppi, Oneto, *Non-defectivity of Segre–Veronese varieties*, Proc. Amer. Math. Soc. Ser. B, 2024.
- Abo, Ottaviani, Peterson, *Induction for secant varieties of Segre varieties*, Trans. Amer. Math. Soc., 2008.
- Abo, Vannieuwenhoven, *Most secant varieties of tangential varieties to Veronese varieties are nondefective*, Trans. Amer. Math. Soc., 2020.
- Alexander, Hirschowitz, *Polynomial interpolation in several variables*, J. Algebraic Geom., 1995.
- Appellof, Davidson, *Strategies for analyzing data from video fluorometric monotoring of liquid chromatographic effluents*, Anal. Chem., 1981.
- Ballico, *On the non-defectivity of Segre–Veronese embeddings*, Math. Zeit., 2024.
- Bernardi, Carlini, Catalisano, Gimigliano, Oneto, *The Hitchhiker Guide to: Secant varieties and tensor decomposition*, Mathematics, 2018.
- Bernardi, Catalisano, Gimigliano, Idà, *Secant varieties to osculating varieties of Veronese embeddings of* $\mathbb{P}^n$, J. Algebra, 2009.
- Blomenhofer, Casarotti, *Nondefectivity of invariant secant varieties*, arXiv, 2024.
- Breiding, Vannieuwenhoven, *The condition number of join decompositions*, SIAM J. Matrix Anal. Appl., 2018.

- Bürgisser, Cucker, *Condition: The Geometry of Numerical Algorithms*, Springer, 2013.
- Casarotti, Mella, *From non-defectivity to identifiability*, J. Eur. Math. Soc., 2023.
- Chiantini, Ottaviani, *On generic identifiability of 3-tensors of small rank*, SIAM J. Matrix Anal. Appl., 2012.
- Chiantini, Ottaviani, Vannieuwenhoven, *An algorithm for generic and low-rank specific identifiability of complex tensors*, SIAM J. Matrix Anal. Appl., 2014.
- Cox, Little, O'Shea, *Ideals, Varieties, and Algorithms*, 5th ed, Springer, 2025.
- De Lathauwer, *Decompositions of a higher-order tensor in block terms—Part II: Definitions and uniqueness*, SIAM J. Matrix Anal. Appl., 2008.
- Greub, *Multilinear Algebra*, Springer, 1978.
- Harris, *Algebraic Geometry: A First Course*, Springer, 1992.
- Hitchcock, *The expression of a tensor or a polyadic as a sum of products*, J. Math. Phys., 1927.
- Jacobsson, Swijsen, Van der Veken, Vannieuwenhoven, *Warped geometries of Segre–Veronese manifolds*, arXiv, 2024.
- Landsberg, *Tensors: Geometry and Applications*, AMS, 2012.
- Lee, *Introduction to Smooth Manifolds*, Springer, 2013.
- Massarenti, Mella, *Bronowski's conjecture and the identifiability of projective varieties*, Duke. Math. J., 2024.

- Oeding, *Hyperdeterminants of polynomials*, Adv. Math., 2016
- Rice, *A theory of condition*, SIAM J. Numer. Anal., 1966.
- Swijsen, Van der Veken, Vannieuwenhoven, *Tensor completion using geodesics on Segre manifolds*, Numer. Linear Algebra Appl., 2022.
- Terracini, Sulle $V_k$ per cui la varietà degli $Sh(h+1)$-secanti ha dimensione minore dell'ordinario, Rend. Circ. Mat. Palermo, 1915.
- Torrance, Vannieuwenhoven, *All secant varieties of the Chow variety are nondefective for cubics and quaternary forms*, Trans. Amer. Math. Soc., 2020.
- Tucker, *Some mathematical notes on three-mode factor analysis*, Psychometrica, 1966.
- Yang, *On partial and generic uniqueness of block term tensor decompositions*, Ann. Univ. Ferrara, 2014.