Nonlinear Rayleigh quotient optimization

Journée scientifique du pôle AM2I: “Introduction à la géométrie des tenseurs”
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Symmetric tensors are homogeneous polynomials

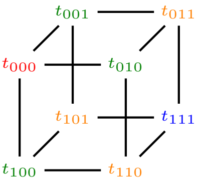
A **symmetric tensor of order** ω over $\mathbb{C}^{n+1}$ is an element of the tensor space

$$\mathrm{Sym}^{\omega}(\mathbb{C}^{n+1})^* \subseteq \underbrace{(\mathbb{C}^{n+1})^* \otimes \cdots \otimes (\mathbb{C}^{n+1})^*}_{\omega \text{ times}} \cong \mathbb{C}^{(n+1)^{\times \omega}}.$$

Its elements are ω-dimensional tensors $T = (t_{i_1 \cdots i_\omega})$ such that

$$t_{i_{\sigma(1)} \cdots i_{\sigma(\omega)}} = t_{i_1 \cdots i_\omega} \text{ for all permutations } \sigma \in S_\omega.$$

For example, a tensor $T \in \mathrm{Sym}^3(\mathbb{C}^2)^* \subseteq (\mathbb{C}^2)^* \otimes (\mathbb{C}^2)^* \otimes (\mathbb{C}^2)^*$ is a cube



with entries t_{000}, $t_{100} = t_{010} = t_{001}$, $t_{110} = t_{101} = t_{011}$, t_{111}.

Symmetric tensors are homogeneous polynomials

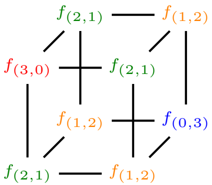
We identify $\mathrm{Sym}^{\omega}(\mathbb{C}^{n+1})^*$ with the space $\mathbb{C}[x]_\omega := \mathbb{C}[x_0, \ldots, x_n]_\omega$ of **homogeneous polynomials of degree** ω in $n+1$ variables. We write

$$f(x_0, \ldots, x_n) = \sum_{|\alpha|=\omega} \binom{\omega}{\alpha} f_\alpha \, x^\alpha \in \mathbb{C}[x]_\omega \,,$$

where $\binom{\omega}{\alpha} = \frac{\omega!}{\alpha_0! \cdots \alpha_n!}$. In particular, we identify f with $(f_\alpha)_{|\alpha|=\omega} \in \mathbb{C}^{\binom{n+\omega}{\omega}}$. For example, we associate the binary cubic in $\mathbb{C}[x_0, x_1]_3 = \mathrm{Sym}^3(\mathbb{C}^2)^*$

$$f(x_0, x_1) = f_{(3,0)} \, x_0^3 + 3 \, f_{(2,1)} \, x_0^2 x_1 + 3 \, f_{(1,2)} \, x_0 x_1^2 + f_{(0,3)} \, x_1^3$$

to the vector $(f_{(3,0)}, f_{(2,1)}, f_{(1,2)}, f_{(0,3)}) \in \mathbb{C}^4$.



Definition (Lim[1], Qi[2])

Consider $\omega \in \mathbb{N}$ and $f \in \mathbb{C}[x]_\omega$. A nonzero vector $\psi \in \mathbb{C}^{n+1}$ is a **(normalized) eigenvector of** f if $\psi \in \mathcal{S}^n = \{x_0^2 + \cdots + x_n^2 = 1\}$ and there exists $\lambda \in \mathbb{C}$ such that

$$\frac{1}{\omega}\nabla f(\psi) = \lambda\,\psi\,.$$

The value $\lambda = f(\psi)$ is the **eigenvalue of** f associated with ψ.
We call **eigenpoint** of f any class $[\psi] \in \mathbb{P}^n$ of an eigenvector of f.

Hence, the normalized eigenvectors ψ of f are the critical points of the **"spherical" Rayleigh optimization problem**

$$\min_{\psi \in \mathbb{C}^{n+1}} f(\psi) \quad \text{subject to } \psi \in \mathcal{S}^n.$$

1. L.-H. Lim. Singular values and eigenvalues of tensors: a variational approach, *in 1st IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, 2005.
2. L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Comput., 2005.

Example: eigenvectors of a binary cubic

Consider the binary cubic ($n = 1$, $\omega = 3$)

$$f(x_0, x_1) = 2\,x_0^3 - 3\,x_0^2 x_1 + 6\,x_0 x_1^2 - x_1^3\,.$$

The eigenpoints of f are the points $[\psi_0 : \psi_1] \in \mathbb{P}^1$ such that the rank of

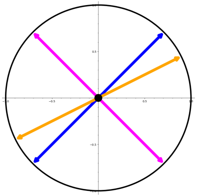
$$A_f(\psi) := \begin{pmatrix} \nabla f(\psi) \\ \psi \end{pmatrix} = \begin{pmatrix} 6(\psi_0^2 - \psi_0\psi_1 + \psi_1^2) & -3(\psi_0 - 4\,\psi_0\psi_1 + \psi_1^2) \\ \psi_0 & \psi_1 \end{pmatrix}$$

is less than two, or equivalently

$$0 = \det A_f(\psi) = (\psi_0 - 2\,\psi_1)(\psi_0 - \psi_1)(\psi_0 + \psi_1)\,.$$

This gives the locus $\{[2:1], [1:1], [1:-1]\} \subseteq \mathbb{P}^1$ of eigenpoints of f, corresponding to the normalized eigenvectors and eigenvalues

$\psi = (\psi_0, \psi_1)$	$\pm(\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5})$	$\pm(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$\pm(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$
$\lambda = f(v)$	$\pm\frac{3\sqrt{5}}{5}$	$\pm\sqrt{2}$	$\pm 3\sqrt{2}$



Number of eigenpoints of a (generic) symmetric tensor, and reality

Theorem (Cartwright-Sturmfels[3], Ottaviani-Oeding[4])

Pick $f \in \mathbb{C}[x]_\omega$ and consider the locus $\mathrm{Eig}(f) \subseteq \mathbb{P}^n$ of eigenpoints of f. If f is generic, then $\mathrm{Eig}(f)$ is zero-dimensional, reduced, and

$$|\,\mathrm{Eig}(f)| = \frac{(\omega-1)^{n+1}-1}{\omega-2} = \sum_{i=0}^{n} (\omega-1)^i\,.$$

Notice that, if $n=1$, then $|\,\mathrm{Eig}(f)| = \omega$ for a generic $f \in \mathbb{C}[x_0,x_1]_\omega$.

Theorem (Kozhasov[5])

It is possible to construct generic homogeneous polynomials $f \in \mathbb{R}[x]_\omega$ with the maximum possible finite number of real eigenpoints.

The case $\omega = 2$ (symmetric matrices) follows from the Spectral Theorem!

3. D. Cartwright and B. Sturmfels, *The number of eigenvalues of a tensor*, Linear Algebra Appl., 2013.
4. L. Oeding and G. Ottaviani, *Eigenvectors of tensors and algorithms for Waring decomposition*, J. Symbolic Comput., 2013.
5. K. Kozhasov, *On fully real eigenconfigurations of tensors*, SIAM J. Appl. Algebra Geom., 2017.

Bombieri-Weyl inner product

Definition

Let $\langle\,,\rangle$ be an inner product on $\mathbb{R}^{n+1}$ and $\omega \in \mathbb{N}$. Fix an orthonormal basis $\{e_0, \dots, e_n\}$ of $\mathbb{R}^{n+1}$ with respect to $\langle\,,\rangle$, and define $x_i = e_i^* \in (\mathbb{R}^{n+1})^*$ for all i. The **Bombieri-Weyl inner product** associated to $\langle\,,\rangle$ and ω is the inner product $\langle\,,\rangle_{\mathrm{BW}}$ on $\mathbb{R}[x]_\omega$ such that

$$\left\{\sqrt{\binom{\omega}{\alpha}}\,x^\alpha\right\}_{|\alpha|=\omega} = \left\{\sqrt{\left(\frac{\omega!}{\alpha_0!\cdots\alpha_n!}\right)}\,x_0^{\alpha_0}\cdots x_n^{\alpha_n}\right\}_{|\alpha|=\omega}$$

is an orthonormal basis of $\mathbb{R}[x]_\omega$.

We will also consider the positive definite quadratic forms $q(x) := \langle x, x\rangle$, $q_{\mathrm{BW}}(f) := \langle f, f\rangle_{\mathrm{BW}}$, and the **isotropic quadrics**

$$\mathcal{Q} := \{[x] \in \mathbb{P}^n \mid q(x) = 0\}\ , \quad \mathcal{Q}_{\mathrm{BW}} := \{[f] \in \mathbb{P}(\mathbb{C}[x]_\omega) \mid q_{\mathrm{BW}}(f) = 0\}\ .$$

in particular $\mathcal{Q}_{\mathrm{BW}}$ is uniquely determined by $\mathcal{Q}$ and ω.

Proposition

Let $\langle\, ,\rangle$ be an inner product on $\mathbb{R}^{n+1}$ and $\omega \in \mathbb{N}$. Let $\langle\, ,\rangle_{\mathrm{BW}}$ be the Bombieri-Weyl inner product on $\mathbb{R}[x]_\omega$ associated with $\langle\, ,\rangle$.
Given $f = (f_\alpha)_{|\alpha|=\omega}$ and $g = (g_\alpha)_{|\alpha|=\omega}$ in $\mathbb{R}[x]_\omega$, written as

$$f(x) = \sum_{|\alpha|=\omega} \binom{\omega}{\alpha} f_\alpha x^\alpha\,, \quad g(x) = \sum_{|\alpha|=\omega} \binom{\omega}{\alpha} g_\alpha x^\alpha\,,$$

we have the identities

$$\langle f, g\rangle_{\mathrm{BW}} = \sum_{|\alpha|=\omega} \binom{\omega}{\alpha} f_\alpha g_\alpha \quad \text{and} \quad q_{\mathrm{BW}}(f) = \sum_{|\alpha|=\omega} \binom{\omega}{\alpha} f_\alpha^2\,.$$

So far we have

1. Introduced symmetric tensors aka homogeneous polynomials
2. Defined eigenvectors of symmetric tensors
3. Equipped the space of real symmetric tensors with the Bombieri-Weyl inner product

In the following, we will use the previous tools to study the **distance function** from a given symmetric tensor, **constrained** to a subset of symmetric tensors.

Before that, we define the **distance degree** of a projective variety.

The Distance Degree of a projective variety

Let $\mathcal{V} \subseteq \mathbb{P}^n$ be a projective variety, $C(\mathcal{V}) \subseteq \mathbb{C}^{n+1}$ its affine cone, and $\mathcal{Q} \subseteq \mathbb{P}^n$ the isotropic quadric associated with a positive-definite quadratic form q on $\mathbb{R}^{n+1}$. Consider the (squared) **distance function from** $u \in \mathbb{C}^{n+1}$

$$d_{q,u}^2 \colon \mathbb{C}^{n+1} \to \mathbb{C}\,, \quad d_{q,u}^2(x) \coloneqq q(u-x)\,.$$

Let $\mathrm{Crit}(d_{q,u}^2, C(\mathcal{V})) \subseteq \mathbb{C}^{n+1}$ be the subset of **critical points of** $d_{q,u}^2$ restricted to the nonsingular locus of $C(\mathcal{V})$. The **distance correspondence** of $(\mathcal{V}, \mathcal{Q})$ is

$$\mathrm{DC}(\mathcal{V}, \mathcal{Q}) \coloneqq \left\{([x],[u]) \mid x \in \mathrm{Crit}(d_{q,u}^2, C(\mathcal{V}))\right\} \subseteq \mathbb{P}_x^n \times \mathbb{P}_u^n\,.$$

Proposition/Definition

The projection $\pi \colon \mathrm{DC}(\mathcal{V}, \mathcal{Q}) \to \mathbb{P}_u^n$ is surjective and generically finite-to-one. The **Distance Degree** of $(\mathcal{V}, \mathcal{Q})$ is

$$\mathrm{DD}(\mathcal{V}, \mathcal{Q}) \coloneqq \deg(\pi) = |\mathrm{Crit}(d_{q,u}^2, C(\mathcal{V}))|\,,\ u \text{ generic.}$$

When q is the standard Euclidean quadratic form, then $\mathrm{DD}(\mathcal{V}, \mathcal{Q})$ is known as the **Euclidean Distance Degree** of $\mathcal{V}$[6].

6. J. Draisma, E. Horobeţ, G. Ottaviani, B. Sturmfels, and R. Thomas, *The Euclidean distance degree of an algebraic variety*, Found. Comput. Math., 2016.

Eigenvectors as critical points of the Bombieri-Weyl distance function

In our case, we consider

1. the projective space $\mathbb{P}(\mathbb{C}[x]_\omega) \cong \mathbb{P}^{\binom{n+\omega}{\omega}-1}$,
2. the Bombieri-Weyl quadratic form q_{BW} on $\mathbb{R}[x]_\omega$,

and define the (squared) **Bombieri-Weyl distance function from** $f \in \mathbb{C}[x]_\omega$

$$d^2_{\mathrm{BW},f} \colon \mathbb{C}[x]_\omega \to \mathbb{C}\,, \quad d^2_{\mathrm{BW},f}(g) \coloneqq q_{\mathrm{BW}}(f-g)\,,$$

which we restrict to the affine cone $C(\mathcal{V}_\omega)$ over the **Veronese variety** $\mathcal{V}_\omega$, that is the image $\nu_\omega(\mathbb{P}^n)$ of the **Veronese embedding**

$$\nu_\omega \colon \mathbb{P}^n \hookrightarrow \mathbb{P}(\mathbb{C}[x]_\omega)\,, \quad \nu_\omega([\psi]) \coloneqq [(\psi^*)^\omega]\,.$$

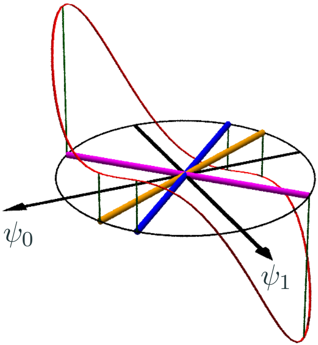
The elements of $C(\mathcal{V}_\omega)$ are also called **rank-one symmetric tensors**.

Proposition

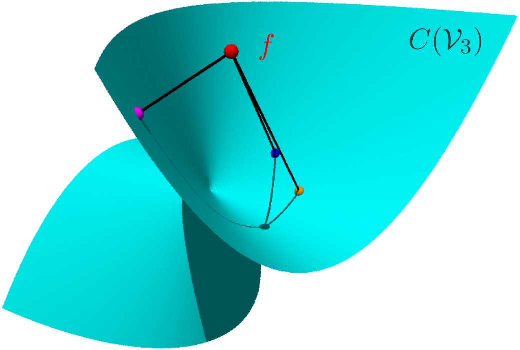
Fix $f \in \mathbb{C}[x]_\omega$. A rank-one symmetric tensor $\lambda(\psi^*)^\omega \in C(\mathcal{V}_\omega)$ belongs to $\mathrm{Crit}(d^2_{\mathrm{BW},f}, C(\mathcal{V}_\omega))$ if and only if ψ is a normalized eigenvector of f with eigenvalue λ.

$\lambda(\psi^*)^\omega \in \mathrm{Crit}(d^2_{\mathrm{BW},f}, C(\mathcal{V}_\omega))$ means that $f - \lambda(\psi^*)^\omega$ is orthogonal (in the BW inner product) to the tangent space of $C(\mathcal{V}_\omega)$ at $(\psi^*)^\omega$:

$$(f - (\psi^*)^\omega) \perp_{\mathrm{BW}} T_{\lambda(\psi^*)^\omega} C(\mathcal{V}_\omega)$$



$f\colon S^1 \to \mathbb{R}$



$f \in \mathbb{R}[x_0, x_1]_3$

In several applications, it is interesting to study the more general **constrained "spherical" Rayleigh optimization problem**

$$\min_{\psi \in \mathbb{C}^{n+1}} f(\psi) \quad \text{subject to } \psi \in \mathcal{S}^n \text{ and } \psi \in C(\mathcal{X}),$$

for some fixed projective variety $\mathcal{X} \subseteq \mathbb{P}^n$. This leads to...

Definition (Salizzoni, S., Weigert[7])

Consider a projective variety $\mathcal{X} \subseteq \mathbb{P}^n$ and $\omega \in \mathbb{N}$.
Given $f \in \mathbb{C}[x]_\omega$, then $\psi \in \mathbb{C}^{n+1}$ is a **(normalized)** $\mathcal{X}$**-eigenvector** of f if $\psi \in \mathrm{Crit}(f, \mathcal{S}^n \cap C(\mathcal{X}))$. The $\mathcal{X}$**-eigenvalue** of f associated with ψ is $\lambda := f(\psi)$. Furthermore, we call $\mathcal{X}$**-eigenpoint** of f any class $[\psi] \in \mathbb{P}^n$ of a normalized $\mathcal{X}$-eigenvector of f.

7. F. Salizzoni, L. Sodomaco, J. Weigert, *Nonlinear Rayleigh quotient optimization*, arXiv:2510.17760, 2025.

The Rayleigh-Ritz degree of a projective variety

Definition (Salizzoni, S., Weigert)

The **Rayleigh-Ritz degree of index ω of $\mathcal{X}$** is the cardinality

$$\mathrm{RRdeg}_{\omega}(\mathcal{X}) := |\{[\psi] \in \mathbb{P}^n \mid [\psi] \text{ is an } \mathcal{X}\text{-eigenpoint of } f\}|$$

for a generic $f \in \mathbb{C}[x]_{\omega}$.

When $\mathcal{X} = \mathbb{P}^n$, the previous facts tell us that

$$\mathrm{RRdeg}_{\omega}(\mathbb{P}^n) = \mathrm{DD}(\mathcal{V}_{\omega}, \mathcal{Q}_{\mathrm{BW}}),$$

where we recall that $\mathcal{V}_{\omega} = \nu_{\omega}(\mathbb{P}^n)$. What about $\mathcal{X} \subsetneq \mathbb{P}^n$?

The Rayleigh-Ritz degree is a particular distance degree

Proposition

Let $\mathcal{Q} \subseteq \mathbb{P}^n$ and $\mathcal{Q}_{\mathrm{BW}} \subseteq \mathbb{P}(\mathbb{C}[x]_\omega)$ be as before. Consider a projective variety $\mathcal{X} \subseteq \mathbb{P}^n$ and its image $\nu_\omega(\mathcal{X}) \subseteq \mathbb{P}(\mathbb{C}[x]_\omega)$ under the Veronese embedding. Fix $f \in \mathbb{C}[x]_\omega$. Then

1. Given $\psi \in \mathcal{S}^n$ and $\lambda \in \mathbb{C}$, then $\lambda(\psi^*)^\omega \in \mathrm{Crit}(d^2_{\mathrm{BW},f}, C(\nu_\omega(\mathcal{X})))$ if and only if $\psi \in \mathrm{Crit}(f, \mathcal{S}^n \cap C(\mathcal{X}))$ and $\lambda = f(\psi)$.
2. Assume that $f \in \mathbb{R}[x]_\omega$ and let $\tilde{\lambda}$ be maximum, in absolute value, among all $\mathcal{X}$-eigenvalues of f whose corresponding $\mathcal{X}$-eigenvector $\tilde{\psi}$ is real. Then $\tilde{\lambda}(\tilde{\psi}^*)^\omega$ is the closest point on $\nu_\omega(\mathcal{X}^{\mathbb{R}})$ to f.

As a consequence, we have the identity

$$\boxed{\mathrm{RRdeg}_\omega(\mathcal{X}) = \mathrm{DD}(\nu_\omega(\mathcal{X}), \mathcal{Q}_{\mathrm{BW}})}$$

namely, the RR degree of $\mathcal{X}$ is a particular distance degree of $\nu_\omega(\mathcal{X})$.

RR degrees of varieties defined implicitly

Theorem (Salizzoni, S., Weigert)

Let $\mathcal{X} \subseteq \mathbb{P}^n$ be a variety of codimension c cut out by polynomials $f_1, \ldots, f_m$ of degrees $\delta_1, \ldots, \delta_m$ such that the first c of them form a regular sequence. Then for all $\omega \in \mathbb{N}$

$$\mathrm{RRdeg}_\omega(\mathcal{X}) \leq \delta_1 \cdots \delta_c \sum_{\substack{i_0+\cdots+i_c=n-c \\ i_0,\ldots,i_c \geq 0}} \prod_{k=1}^{c} (\delta_k - 1)^{i_k} \cdot \sum_{\ell=0}^{i_0} (\omega - 1)^\ell \,.$$

The equality holds if $\mathcal{X}$ is a generic complete intersection.

Corollary

Let $\mathcal{X} \subseteq \mathbb{P}^n$ be a generic hypersurface of degree δ. Then for all $\omega \geq 1$

$$\mathrm{RRdeg}_\omega(\mathcal{X}) = \delta \sum_{i=0}^{n-1} (\delta - 1)^{n-1-i} \sum_{\ell=0}^{i} (\omega - 1)^\ell \,.$$

Example: RR degree of the Fermat cubic

Consider the curve $\mathcal{X} \subseteq \mathbb{P}^2$ cut out by the Fermat cubic $f_1 = x_0^3 + x_1^3 + x_2^3$, and let $q = x_0^2 + x_1^2 + x_2^2$ be the standard Euclidean quadratic form.
We expect to have

$$\mathrm{RRdeg}_2(\mathcal{X}) = 3\sum_{i=0}^{1} 2^{1-i}(i+1) = 3(2+2) = 12\,. \qquad (*)$$

In fact, given a generic $f \in \mathbb{C}[x_0, x_1, x_2]_2$, the $\mathcal{X}$-eigenpoints of f are the points of intersection between the curves $\mathcal{X}$ and $\mathbb{V}(h)$, where

$$h(x) := \det\begin{pmatrix} x_0 & x_1 & x_2 \\ \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f}{\partial x_0} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix} = 3 \cdot \det\begin{pmatrix} x_0 & x_1 & x_2 \\ x_0^2 & x_1^2 & x_2^2 \\ \frac{\partial f}{\partial x_0} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix}\,.$$

Therefore, if the curves defined respectively by f and h are in general position, by Bézout's theorem the number of $\mathcal{X}$-eigenpoints of f is $3 \cdot 4 = 12$, as predicted in $(*)$.

RR degrees of varieties defined parametrically

Theorem (Salizzoni, S., Weigert)

Consider two positive integers $n > m$. Let $g_0, \ldots, g_n$ be $n+1$ generic homogeneous polynomials of degree d in $m+1$ variables, and assume that the image $\mathcal{X}$ of the morphism

$$g\colon \mathbb{P}^m \to \mathbb{P}^n\,, \quad [\psi] \mapsto [g_0(\psi) : \cdots : g_n(\psi)]$$

associated with the polynomials g_i is nonsingular. Then

$$\mathrm{RRdeg}_\omega(\mathcal{X}) = \begin{cases} (m+1)(2d-1)^m & \text{if } \omega = 2 \\ \frac{(\omega d-1)^{m+1}-(2d-1)^{m+1}}{d(\omega-2)} & \text{if } \omega > 2. \end{cases}$$

RR degrees of varieties in general position

Definition

Consider an embedded nonsingular projective variety $\mathcal{Y} \hookrightarrow \mathbb{P}^n$ of dimension m, via the line bundle $\mathcal{L}$ with $L = c_1(\mathcal{L})$.
The **Generic Distance Degree** of $\mathcal{Y}$ is

$$\mathrm{gDD}(\mathcal{Y}) = \sum_{i=0}^{m} (-1)^i (2^{m+1-i} - 1) \int_{\mathcal{Y}} c_i(\mathcal{Y}) \cdot L^{m-i} \,.$$

In particular $\mathrm{DD}(\mathcal{Y}, \mathcal{Q}) \leq \mathrm{gDD}(\mathcal{Y})$ for any isotropic quadric $\mathcal{Q} \subseteq \mathbb{P}^n$ and the equality is attained for a generic $\mathcal{Q}$.

Theorem (Salizzoni, S., Weigert)

Let $\mathcal{X} \subseteq \mathbb{P}^n$ be a nonsingular variety of dimension m, in general position with respect to the fixed isotropic quadric $\mathcal{Q} \subseteq \mathbb{P}^n$. Then

$$\mathrm{RRdeg}_\omega(\mathcal{X}) = \mathrm{gDD}(\nu_\omega(\mathcal{X})) - \underbrace{(\omega - 1)\mathrm{gDD}(\nu_\omega(\mathcal{X} \cap \mathcal{Q}))}_{\text{“defect”}} \,.$$

Furthermore, if the embedding of $\mathcal{X} \hookrightarrow \mathbb{P}^n$ is given by a line bundle $\mathcal{L}$ with first Chern class $L = c_1(\mathcal{L})$, then

$$\mathrm{RRdeg}_\omega(\mathcal{X}) = \sum_{i=0}^{m} (-1)^i \left(\sum_{j=0}^{m-i} \omega^{m-i-j} 2^j \right) \int_{\mathcal{X}} c_i \cdot L^{m-i} .$$

Example

$$\mathrm{RRdeg}_\omega(\mathcal{X}) = \begin{cases} \int_X (\omega+2)L - c_1 & \text{if } m = 1 \\ \int_X (\omega^2+2\omega+4)L^2 - (\omega+2)c_1 \cdot L + c_2 & \text{if } m = 2. \end{cases}$$

One may use these formulas for any nonsingular projective variety, as long as its Chern classes are known. For example, all nonsingular toric varieties.

A well-known nonsingular toric embedding is the **Segre-Veronese embedding** of a product of k projective spaces $\mathbb{P}^{\boldsymbol{m}} := \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_k}$ in $\mathbb{P}(V)$, where $V := \otimes_{i=1}^k \mathbb{C}[x_i]_{d_i}$ is the space of k-homogeneous polynomials of multidegree $\boldsymbol{d} = (d_1, \ldots, d_k)$:

$$\nu_{\boldsymbol{d}} \colon \mathbb{P}^{\boldsymbol{m}} \hookrightarrow \mathbb{P}(V)\,, \quad \nu_{\boldsymbol{d}}([\psi_1], \ldots, [\psi_k]) := [(\psi_1^*)^{d_1} \otimes \cdots \otimes (\psi_k^*)^{d_k}]\,.$$

The **Segre-Veronese variety** is $\mathcal{SV}_{\boldsymbol{d}} := \nu_{\boldsymbol{d}}(\mathbb{P}^{\boldsymbol{m}})$. Its elements are partially symmetric tensors of rank one.

We computed $\mathrm{RRdeg}_\omega(\mathcal{SV}_{\boldsymbol{d}})$ in two cases, for any $(\omega, \boldsymbol{m}, \boldsymbol{d})$:

1. when $\mathcal{SV}_{\boldsymbol{d}}$ is in general position with respect to $\mathcal{Q} \subseteq \mathbb{P}(V)$,
2. when $\mathcal{Q} = \mathcal{Q}_{\mathrm{BW}}$ is the isotropic quadric associated to the Bombieri-Weyl inner product in $V^{\mathbb{R}} = \otimes_{i=1}^k \mathbb{R}[x_i]_{d_i}$ for any $k \geq 1$, for any fixed inner products in the factors $\mathbb{R}[x_i]_{d_i}$.

For example, if $\boldsymbol{m} = \boldsymbol{d} = (1, \dots, 1)$, the variety $\mathcal{SV}_{\boldsymbol{d}}$ encodes k-dimensional binary tensors of $V \cong (\mathbb{C}^2)^{\otimes k}$ of rank one. These correspond to "separable quantum states", in contrast to "entangled quantum states" of higher rank.

Corollary

Let $\boldsymbol{m} = \boldsymbol{d} = (1, \dots, 1)$ and assume that $\mathcal{X} = \mathcal{SV}_{\boldsymbol{d}}$ is in general position with respect to $\mathcal{Q}$. Then

$$\mathrm{RRdeg}_\omega(\mathcal{X}) = \begin{cases} k!\, 2^k \sum_{i=0}^k \frac{(-1)^i}{i!}(k+1-i) & \text{if } \omega = 2 \\ k! \sum_{i=0}^k \frac{(-1)^i}{i!} \frac{\omega^{k+1-i} 2^i - 2^{k+1}}{\omega - 2} & \text{if } \omega \neq 2. \end{cases}$$

If instead we choose $\mathcal{Q} = \mathcal{Q}_{\mathrm{BW}}$, then $\mathrm{RRdeg}_\omega(\mathcal{X})$ is much smaller:

Proposition

Let $\boldsymbol{m} = \boldsymbol{d} = (1, \dots, 1)$ and consider $\mathcal{X} = \mathcal{SV}_{\boldsymbol{d}} \subseteq \mathbb{P}((\mathbb{C}^2)^{\otimes k})$. Fix the Bombieri-Weyl isotropic quadric $\mathcal{Q}_{\mathrm{BW}} \subseteq \mathbb{P}((\mathbb{C}^2)^{\otimes k})$. Then

$$\mathrm{RRdeg}_\omega(\mathcal{X}) = k!$$

Let's recall this result mentioned before:

Theorem (Kozhasov[8])

It is possible to construct generic homogeneous polynomials $f \in \mathbb{R}[x]_\omega$ with the maximum possible finite number of real eigenpoints.

This property is no longer true for real $\mathcal{X}$-eigenpoints in general.

For example, consider the plane conic $\mathcal{X} = \mathbb{V}(x_1^2 - x_0x_2) \subseteq \mathbb{P}^2$ and the standard Euclidean quadric $\mathcal{Q} = \mathbb{V}(x_0^2 + x_1^2 + x_2^2)$.

We verified that **every** real $f \in \mathbb{C}[x]_2$ with finitely many $\mathcal{X}$-eigenpoints has either 2 or 4 real $\mathcal{X}$-eigenpoints.

In particular, f has always **fewer** than $\mathrm{RRdeg}_2(\mathcal{X}) = 6$ real $\mathcal{X}$-eigenpoints.

8. K. Kozhasov, *On fully real eigenconfigurations of tensors*, SIAM J. Appl. Algebra Geom., 2017.

Merci!
Thank you!

https://arxiv.org/abs/2510.17760