# Terracini-inspired insights for tensor decomposition

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# A geometric observation

Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety. Consider $r$ **general** points $p_1, \ldots, p_r$ of $X$. Then, **we expect**

$\dim\langle T_{p_1}X, \ldots, T_{p_r}X\rangle$ **to be** $\min\{N, r(\dim X + 1) - 1\} = \mathit{expdim}$.

When the above *expdim* is not reached, $X$ is quite special!

# Terracini's (first) Lemma

The ***r*-secant variety** $\sigma_r(X)$ of $X \subset \mathbb{P}^N$ is

$$\sigma_r(X) := \overline{\bigcup_{p_1,\ldots,p_r \in X} \langle p_1, \ldots, p_r \rangle} \subset \mathbb{P}^N.$$

$$X = \sigma_1(X) \subset \sigma_2(X) \subset \cdots \subset \sigma_{g_r}(X) \cong \mathbb{P}^N$$

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### Lemma (Terracini)

*For a general $T \in \langle p_1, \ldots, p_r \rangle$ where $p_i \in X \subset \mathbb{P}^N$ are general,*

$$\dim T_T\sigma_r(X) = \dim\langle T_{p_1}X, \ldots, T_{p_r}X \rangle.$$

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$$\dim T_T \sigma_r(X) = \dim \langle T_{p_1} X, \ldots, T_{p_r} X \rangle.$$

$X$ is ***r*-defective** when

$$\dim \sigma_r(X) < \min\{N, r(\dim X + 1) - 1\}.$$

# A bit of context on secant varieties

## to understand the importance of Terracini's Lemma

Given $X \subset \mathbb{P}^N$,
the **$X$-rank** of a point $T \in \mathbb{P}^N$ is the minimum integer $r$ such that

$$T \in \langle p_1, \ldots, p_r \rangle, \text{ for distinct } p_i \in X.$$

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the generic element of $\sigma_r(X)$ has $X$-rank $r$.

Knowing the dimension of $\sigma_r(X)$ is useful to understand the generic $X$-rank.

# Why am I telling you all this?

In some special cases, $X$ is actually a **tensor variety**:

$$X = \begin{cases} k\text{-factor Segre} & \{v_1 \otimes \cdots \otimes v_k\} \subset \mathbb{P} V_1 \otimes \cdots \otimes V_k \\ \text{d-Veronese} & \{v^{\otimes d}\} \subset \mathbb{P}\mathrm{Sym}^d V \\ \text{Segre-Veronese} & \{v_1^{\otimes d_1} \otimes \cdots \otimes v_k^{\otimes d_k}\} \subset \mathbb{P} \bigotimes_{i=1}^k \mathrm{Sym}^{d_i} V_i \\ \text{Grassmannians} & \{v_1 \wedge \cdots \wedge w_k\} \subset \mathbb{P} \wedge^k V \\ \cdots \end{cases}$$

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and the $X$-rank is actually a **tensor rank**

$$\min r \text{ such that } T = \begin{cases} \sum_{i=1}^r v_{i,1} \otimes \cdots \otimes v_{i,k} & \text{rank} \\ \sum_{i=1}^r v_i^{\otimes d} & \text{Waring or sym rank} \\ \sum_{i=1}^r v_{i,1}^{\otimes d_1} \otimes \cdots \otimes v_{i,k}^{\otimes d_k} & \text{partially sym rank} \\ \sum_{i=1}^r v_{i,1} \wedge \cdots \wedge w_{i,k} & \text{skew-sym rank} \\ \ldots \end{cases}$$

# Let's go back to our geometric observation...

When $X \subset \mathbb{P}^N$ is not $r$-defective, for general points $p_1, \ldots, p_r$ we have $\dim\langle T_{p_1}X, \ldots, T_{p_r}X\rangle = \min\{N, r(\dim X + 1) - 1\}$.

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However, there may exist **special points** $p_1, \ldots, p_r \in X_{sm}$ for which

$$\dim\langle T_{p_1}X, \ldots, T_{p_r}X\rangle < \min\{N, r(\dim X + 1) - 1\}.$$

**We are interested in these points.**

# ...and let's make an example!

Let $\dim V = n+1$, $X = \nu_d(\mathbb{P}(V)) = \{[v^d], v \in V\}$, so the image of the $d$ Veronese map $\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ such that $[v] \mapsto [v^d]$.

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Fix $d = 3$, $\dim V = 3$.

- For $p_1 = e_0^3$, a basis of $\hat{T}_{p_1}X$ is $e_0^3, \quad e_0^2 e_1, \quad e_0^2 e_2$.
- For $p_2 = e_1^3$, a basis of $\hat{T}_{p_2}X$ is $e_1^2 e_0, \quad e_1^3, \quad e_1^2 e_2$.
- For $p_3 = (e_0 + e_1)^3$, a basis of $\hat{T}_{p_3}X$ is $(e_0+e_1)^3, \quad (e_0+e_1)^2(e_0-e_1), \quad (e_0+e_1)^2 e_2$.

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But a basis of $\langle \hat{T}_{p_1}X, \hat{T}_{p_2}X, \hat{T}_{p_3}X\rangle$ is $e_0^3, \quad e_0^2e_1, \quad e_0^2e_2, \quad e_1^2e_0, \quad e_1^3e_1, \quad e_1^2e_2, \quad (e_0+e_1)^2e_2$.

## How to interpret the problem

Take $X = \nu_d(\mathbb{P}(V))$, $p = v^d$, so $\hat{T}_p X = \{v^{d-1}w, w \in V\}$.

- $I_p$ contains all hypersurfaces passing through $v^d$
- $(I_p)^2$ contains all hypersurfaces singular at $v^d$
- the 0-dim scheme defined by $(I_p)^2$ is the **double point** $2p$

Then (Lasker)

$$\boxed{\hat{T}_p X^\vee = (I_p^2)_d}$$

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**Ex:** $p = e_0^d$, $I_p = (x_1, \ldots, x_n)$, $I_p^2 = (x_1^2, x_1x_2, \ldots, x_n^2)$,

$(I_p^2)_d$ is given by all degree $d$ monomials **but** $x_0^d, x_0^{d-1}x_1, \ldots, x_0^{d-1}x_n$

while

$$\hat{T}_p X = \langle e_0^d, e_0^{d-1}e_1, \ldots, e_0^{d-1}e_n \rangle.$$

# How to interpret the problem II

If we take $r$-points $A = \{p_1, \ldots, p_r\} \subset X_{sm}$ and call

- $2A = \{2p_1, \ldots, 2p_r\}$, $I_{2A} = \cap_i (I_{p_i})^2$
- $\langle 2A \rangle = \langle T_{p_1}X, \ldots, T_{p_r}X \rangle$.

Then

$$\operatorname{codim}\langle 2A \rangle = h^0(\mathcal{I}_{2A,\mathbb{P}^n}(d))$$

and

**understanding $\dim\langle 2A \rangle$ is now an interpolation problem.**

This is true for $X =$ **tensor variety**.

# Terracini Locus

[BBS] and [BC]

Let $X \subset \mathbb{P}^N$ be a non-degenerate irreducible variety embedded via an ample line bundle. For a set $A = \{p_1, \ldots, p_r\} \subset X_{sm}$ of $r$ points let

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The ***r*-Terracini locus** $\mathbb{T}_r(X)$ of $X$ is

$$\begin{aligned}\mathbb{T}_r(X) &= \{A \subset X_{sm} \mid \langle 2A \rangle \neq \mathbb{P}^N \text{ and } \dim\langle 2A \rangle < r(\dim X + 1) - 1\} \\ &= \{A \subset X_{sm} \mid h^0(\mathcal{I}_{2A}(1)) \cdot h^1(\mathcal{I}_{2A}(1)) \neq 0\}.\end{aligned}$$

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It is not necessary to consider all values of $r$.

- $\mathbb{T}_1(X) = \emptyset$,
- If $A \in \mathbb{T}_r(X)$ then $A \cup \{p\} \in \mathbb{T}_{r+1}(X)$ for all $p \in X_{sm}$.

## Geometric interpretation

The **abstract *r*-th secant variety** of $X \subset \mathbb{P}^N$ is

$$Abs_r(X) := \{(T; (p_1, \ldots, p_r)) \in \mathbb{P}^N \times X^r \colon T \in \langle p_1, \ldots, p_r \rangle\}.$$

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$$Abs_r(X) := \{(T; (p_1, \ldots, p_r)) \in \mathbb{P}^N \times X^r : T \in \langle p_1, \ldots, p_r \rangle\}.$$

The first projection

$$T_r : Abs_r(X) \to \bigcup_{p_1, \ldots, p_r \in X} \langle p_1, \ldots, p_r \rangle$$

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**The $r$-th Terracini locus measures the degeneracy of the differential of this map.**

Why is this relevant?

# 1) The condition number of a tensor decomposition

Keep in mind Nick's talk

Recall the Terracini map $T_r : Abs_r(X) \to \sigma_r^0(X)$.
The **condition number** of $p_1, \ldots, p_r$ is

$$\kappa(p_1, \ldots, p_r) := \begin{cases} \|(\mathrm{d}T_{r,(p_1,\ldots,p_r)})^{-1}\|_2 & \text{if } \mathrm{d}T_r \text{ is invertible at } p_1, \ldots, p_r, \\ \infty & \textit{otherwise}. \end{cases}$$

- Call $T^{-1}_{r,(p_1,\ldots,p_r)}$ the local inverse of $T_r$ at $p_1, \ldots, p_r$.
- If the differential $\mathrm{d}T_r$ of $T_r$ at $p_1, \ldots, p_r$ is invertible then a local inverse exists.

# Terracini loci and condition numbers

$$\kappa(p_1, \dots, p_r) := \begin{cases} \|(\mathrm{d}T_{r,(p_1,\dots,p_r)})^{-1}\|_2 & \text{if } \mathrm{d}T_r \text{ is inv. at } p_1, \dots, p_r \\ \infty & \textit{otherwise}. \end{cases}$$

**Tuples of the Terracini loci correspond to tuples with an infinite condition number**

The Terracini locus embodies tuples that have a **bad** behaviour and we want to avoid them wen performing a tensor decomposition!

# 2) Identifiability

A point $T \in \mathbb{P}^N$ of $X$-rank $r$ is **identifiable** if

$$T = \sum_{i=1}^{r} p_i, \text{ where } p_i \in X$$

in a **unique** way.

There are many results for identifiability of generic tensors.

## Terracini loci and the identifiability quest

Identifiability for specific tensors:

- Completely solved:
  - Binary forms (Sylvester)
  - Identifiability of non-structured rank-3 tensors [BBS]
- Many results on forms of low degree (Chiantini et al.)
- Criterion: Kruskal, many generalizations of Kruskal's ([COV],[LP],...)

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However, given $T \in \mathbb{P}^N$ with

$$T = \sum_{i=1}^{r} p_i,$$

**if we want even a slight chance that $T$ is identifiable then $p_1, \ldots, p_r$ must lie outside the Terracini locus**.

## Selected results on Terracini loci

Many people have worked on Terracini loci recently

- It has been introduced in [BC] and [BBS].
- The work [LM] characterizes emptyness in the case of toric varieties.
- In [GSTT] we characterize the first and second non-empty Terracini locus in the case of Veronese and Segre-Veronese varieties.

There is still much work to do!

# What about the second Terracini's Lemma?

**Lemma (Second Terracini's Lemma)**

*Let $p_1, \ldots, p_r \in X$ be general points and assume that $X$ is $r$-defective. Then, there is a positive dimensional variety $C \subseteq X$ through $p_1, \ldots, p_r$ such that*

$$\textit{if } p \in C \textit{ then } T_pX \subseteq \langle T_{p_1}X, \ldots, T_{p_r}X \rangle.$$

The second Terracini's Lemma led to

- **weak defectivity** [Chiantini-Ciliberto]
- **tangential weak defectivity** [Chiantini-Ottaviani] $\sim$ there is a positive dimensional subvariety $C$ of $X$ for which $T_q C \subset \langle 2A \rangle$ for general $A \subset X$.

## A geometric question II

### What happens in the non generic scenario?

When $X \subset \mathbb{P}^N$ is not $r$-twd, for general points $p_1, \dots, p_r$ we have that if $\langle T_{p_1}X, \dots, T_{p_r}X \rangle \supset T_P X \neq \emptyset$ then $P = p_i$ for some $i$.

However, there might exists **special** sets of **points** $A = \{p_1, \dots, p_r\} \subset X_{sm}$ for which

$$\langle 2A \rangle \supset T_q X, \text{ for infinitely many } q \in X.$$

**We are interested in these points.**

## A new geometric object

Let $X \subset \mathbb{P}^N$ be an integral and non-degenerate variety embedded via an ample line bundle. We study

$$\begin{aligned}\mathcal{E}_r(X) &= \{A \subset X_{sm} \mid \langle 2A \rangle \neq \mathbb{P}^N, \langle 2A \cup \{2p\} \rangle = \langle 2A \rangle, \text{ for } p \in X_{sm} \setminus A\} \\ &= \{A \subset X_{sm} \mid h^0(\mathcal{I}_{2A}(1)) \neq 0, h^0(\mathcal{I}_{2A \cup 2p}(1)) = h^0(\mathcal{I}_{2A}(1))\}\end{aligned}$$

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For a given set $A \subset X_{sm}$, the **tangential contact locus** $C(A)$ of $A$ is

$$C(A) = \{p \in X_{sm} \setminus A \mid \langle 2p \rangle = T_pX \subset \langle 2A \rangle\}$$

and we are esepecially interested when $C(A)$ is **infinite**.

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$\mathcal{E}_r(X)$ is an exploration of **non-generic** set of **points having a positive dimensional tangential contact locus**.

# Relation between $\mathbb{T}_r(X)$ and $\mathcal{E}_r(X)$

Is there any?

- A curve $C \subset \mathbb{P}^2$ of degree 4 has 28 bitangents. So $\mathbb{T}_2(C) \neq \emptyset$. But each bitangent does not intersect $C$ in other points $\implies \mathcal{E}_2(X) = \emptyset$.

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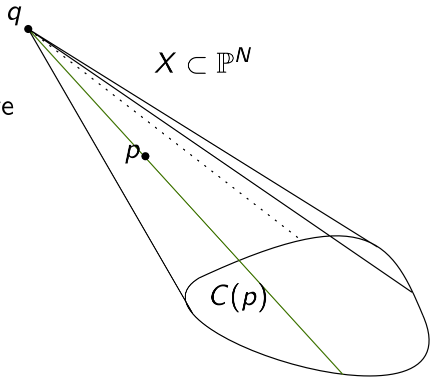
- For a cone $X \subset \mathbb{P}^N$, $\mathbb{T}_1(X) = \emptyset$ while $\mathcal{E}_1(X) \neq \emptyset$ and the contact locus is positive dimensional.



# What happens for tensor related varieties?

- For rational normal curves one easily shows that it is always empty
- Veronese and Segre-Veronese varieties: **work in progress**

Why is this relevant?

# 1) Identifiability

[Proposition 2.4, CO] shows that

If there exists a set $A$ of $r$ particular points such that the span $\langle 2A \rangle$ contains $T_pX$ only if $p \in A$ then identifiability for general rank-$r$ tensors holds.

$$\Downarrow$$

if $\mathcal{E}_r(X) \neq X^r$ then generic $r$-identifiability holds

In the quest for identifiability one wants to **avoid both** $\mathbb{T}_r(X)$ **and** $\mathcal{E}_r(X)$.

## 2) Other applications?

Possibly, TBD :)

Thank you!