Low rank approximation of moment sequences and tensors

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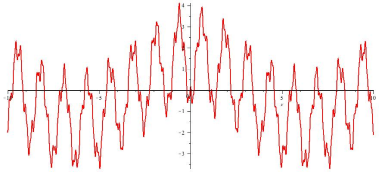
November 26-27, 2024

Reconstruction of signals

Given a function or signal $f(t)$:



decompose it as

$$f(t) = \sum_{i=1}^{r'} (a_i cos(\mu_i t) + b_i sin(\mu_i t)) e^{\nu_i t} = \sum_{i=1}^{r} \omega_i e^{\zeta_i t}$$

☞ Compute the values $\sigma_0 = f(0), \sigma_1 = f(1), \ldots$ and deduce the decomposition from this sequence (Gaspard Baron de Prony)

Prony's method (1795)

For the signal $f(t) = \sum_{i=1}^{\mathrm{r}} \omega_{\mathrm{i}} e^{\zeta_{\mathrm{i}} t}$, $(\omega_i, \zeta_i \in \mathbb{C})$,

- Evaluate f at $2\,r$ regularly spaced points: $\sigma_0 := f(0), \sigma_1 := f(1), \ldots$
- Compute a non-zero element $\mathrm{p} = [\mathrm{p}_0, \ldots, \mathrm{p}_{\mathrm{r}}]$ in the kernel:

$$\left[\begin{array}{ccc|c} \sigma_0 & \sigma_1 & \cdots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{r-1} & \cdots & \sigma_{2r-1} & \sigma_{2r-1} \end{array}\right] \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

- Compute the roots $\xi_1 = \mathrm{e}^{\zeta_1}, \ldots, \xi_{\mathrm{r}} = \mathrm{e}^{\zeta_{\mathrm{r}}}$ of $p(x) := \sum_{i=0}^{r} p_i x^i$.
  (or the generalised eigenvalues of H_0, H_1) to recover the frequencies ζ_{i}.
- Solve the system

$$\begin{bmatrix} 1 & \cdots & \cdots & 1 \\ \xi_1 & & & \xi_r \\ \vdots & & & \vdots \\ \xi_1^{r-1} & \cdots & \cdots & \xi_r^{r-1} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{\mathrm{r}} \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_{r-1} \end{bmatrix}.$$

  to recover the weights ω_{i}.

Blind identification






Observing $\boldsymbol{y}(t)$ with

$$\boldsymbol{y}(t) = H\ \boldsymbol{s}(t)$$

☞ **find H and** s(t)

- If the sources are statistically independent, using the high order statistics $\mathbb{E}(y_i\, y_j\, y_k \cdots)$ of the signal $\boldsymbol{y}(t)$, **decompose the symmetric tensor** $T = \sum_{i,j,k,\dots} \mathbb{E}(y_i\, y_j\, y_k \cdots)\, x_i\, x_j\, x_k \cdots = \sum_{|\alpha|=d} \binom{d}{\alpha} \mathbb{E}(\boldsymbol{y}^\alpha)\boldsymbol{x}^\alpha$ as
$$T(\boldsymbol{x}) = \sum_{i=1}^{r} (H_i, \boldsymbol{x})^d$$
- Deduce the geometry of the sources $H = [H_1, \dots, H_r]$ and $\boldsymbol{s}(t)$.

Symmetric tensor decomposition and Waring problem (1770)

Symmetric tensor decomposition problem:

Given a homogeneous polynomial $T(\boldsymbol{x}) \in S^d(\mathbb{K}^n)$ of degree d in the variables $\boldsymbol{x} = (x_1, \ldots, x_n)$ with coefficients $\in \mathbb{K}$:

$$T(\boldsymbol{x}) = \sum_{|\alpha|=d} T_\alpha \, \boldsymbol{x}^\alpha,$$

find a minimal decomposition of T of the form

$$T(\boldsymbol{x}) = \sum_{i=1}^{r} \omega_i (\xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)^d = \sum_{i=1}^{r} \omega_i (\xi_i, \boldsymbol{x})^d$$

with $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n}) \in \overline{\mathbb{K}}^n$ spanning distinct lines, $\omega_i \in \overline{\mathbb{K}}$ (equivalently $T = \sum_{i=1}^r \omega_i \, \xi_i^{\otimes d}$).

The minimal r in such a decomposition is called the **rank** of T.

Tensor decomposition as a moment problem

Apolar product: For $T = \sum_{|\alpha|=d} t_\alpha \boldsymbol{x}^\alpha$, $T' = \sum_{|\alpha|=d} t'_\alpha \boldsymbol{x}^\alpha \in S^d(\mathbb{K}^n)$,

$$\langle T, T' \rangle_d = \sum_{|\alpha|=d} t_\alpha \, t'_\alpha \binom{d}{\alpha}^{-1}.$$

Property: $\langle T, (\xi, \boldsymbol{x})^d \rangle = T(\xi)$

Let

$$\begin{aligned} T^* : S^d &\rightarrow \mathbb{K} \\ p &\mapsto \langle T, p \rangle_d \end{aligned}$$

T^* is a **linear functional** given by its (pseudo) **moments** $T^*(\boldsymbol{x}^\alpha) = t_\alpha \binom{d}{\alpha}^{-1}$.

Theorem (Weighted Sum of Diracs (WSD))

$$T(\boldsymbol{x}) = \sum_{i=1}^{r} \omega_i (\xi_i, \boldsymbol{x})^d \Leftrightarrow T^* = \sum_{i=1}^{r} \omega_i \delta_{\xi_i}.$$

Cubature formula

For a (positive Borel) measure μ on $\mathbb{R}^n$ with compact support,

- $\sigma_\alpha = \int \boldsymbol{y}^\alpha \mu(dy)$ is called the **moment** of $\boldsymbol{y}^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$.

Theorem (Tchakaloff, 1957)

For μ with compact support and $d \in \mathbb{N}$,

$$T(\boldsymbol{x}) = \int (1 + (\boldsymbol{x}, \boldsymbol{y}))^d \mu(d\boldsymbol{y}) = \sum_{|\alpha| \leq d} \binom{d}{\alpha} \sigma_\alpha \boldsymbol{x}^\alpha$$

has a decomposition of the form

$$T(\boldsymbol{x}) = \sum_{i=1}^{r} \omega_i (1 + (\xi_i, \boldsymbol{x}))^d \Leftrightarrow T^* = \sum_{i=1}^{r} \omega_i \delta_{\xi_i}.$$

with $\xi_i \in \operatorname{supp}(\mu)$, $\omega_i > 0$.

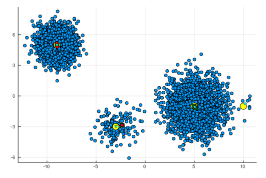
☞ $\mu \sim \sum_{i=1}^{r} \omega_i \delta_{\xi_i}$ on $\mathbb{R}[\boldsymbol{x}]_{\leq d}$ $\Rightarrow$ cubature formulae.

Gaussian mixtures

A mixture of (spherical) Gaussian distributions

$$g(\mathrm{y}) = \sum_{k=1}^{r} \omega_k f(\mathrm{y}, \mu_k, \sigma_k)$$

where



- $f(\mathrm{y}, \mu_k, \sigma_k)$ is the normal distribution of mean $\mu_k \in \mathbb{R}^n$ and covariance $\Sigma_k = \mathrm{diag}(\sigma_k^2) \in \mathbb{R}^{n \times n}$,
- ω_k is the proportion of mixture of the k^{th} normal distribution $f(\mathrm{y}, \mu_k, \sigma_k)$.

Theorem

For $\overline{\sigma}$ the smallest eigenvalue of $\mathbb{E}[\mathrm{y} \otimes \mathrm{y}] - \mathbb{E}[\mathrm{y}] \otimes \mathbb{E}[\mathrm{y}]$ and v its unit eigenvector,

- $M_1(\mathrm{x}) := \mathbb{E}[\langle \mathrm{v}, \mathrm{y} - \mathbb{E}[\mathrm{y}] \rangle^2 (\mathrm{y} \cdot \mathrm{x})] = \sum_k \omega_k \, \sigma_k^2 \, (\mu_k \cdot \mathrm{x})$
- $M_2(\mathrm{x}) := \mathbb{E}[(\mathrm{y} \cdot \mathrm{x})^2] - \overline{\sigma} \, \|\mathrm{x}\|^2 = \sum_k \omega_k \, (\mu_k \cdot \mathrm{x})^2$
- $M_3(\mathrm{x}) := \mathbb{E}[(\mathrm{y} \cdot \mathrm{x})^3] - 3 M_1(\mathrm{x}) \, \|\mathrm{x}\|^2 = \sum_k \omega_k \, (\mu_k \cdot \mathrm{x})^3$

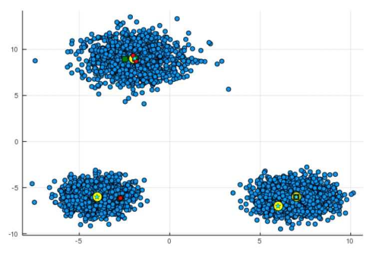
[Hsu, Kakade 2013]

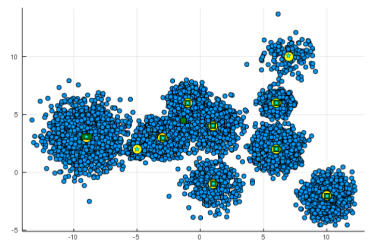
Expectation Maximisation (EM):

$$\max \sum_{i=1}^{p} \log(\sum_{k=1}^{r} \omega_k f(x_i, \mu_k, \sigma_k))$$

by alternate iterative optimization from an initial start.

Comparaison with k-means, split and tensor decomposition:

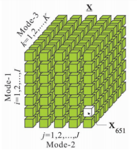




Examples with $n = 6, r = 4$; $n = 30, r = 10$

[Joint work with Rima Khouja, using Julia package `TensorDec.jl`]

Multilinear tensors

$T = (t_{i_1,i_2,i_3}) \in E_1 \otimes E_2 \otimes E_3 \equiv$ 

Definition: $\langle T, T' \rangle = \sum_{i_1,i_2,i_3} t_{i_1,i_2,i_3}\, t'_{i_1,i_2,i_3}$

Let
$$\begin{aligned} T^* : E_1 \otimes E_2 \otimes E_3 &\to \mathbb{K} \\ T' &\mapsto \langle T, T' \rangle \end{aligned}$$

$T^* \in E_1^* \otimes E_2^* \otimes E_3^*$ is a linear functional given by its (pseudo) moments $T^*(x_{1,i_1} x_{2,i_2} x_{3,i_3}) = t_{i_1,i_2,i_3}$.

Theorem (WSD)

$$T = \sum_{i=1}^{r} \omega_i\, \boldsymbol{u}_i \otimes \boldsymbol{v}_i \otimes \boldsymbol{w}_i \Leftrightarrow T^* = \sum_{i=1}^{r} \omega_i\, \delta_{\boldsymbol{u}_i} \otimes \delta_{\boldsymbol{v}_i} \otimes \delta_{\boldsymbol{w}_i}$$

Decomposition methods

1. Associate **linear operators/matrices** to the tensor.

2. Recover the decomposition from the **image, kernel, eigenspaces** of the operators.

Flattening for multilinear tensors

For a multilinear tensor $T = [t_{i_1,\dots,i_l}] \in \mathbb{K}^{n_1 \times \cdots \times n_l}$, **flattening** or **matricisation** in mode $(n_1 \times \cdots \times n_k, n_{k+1} \times \cdots \times n_l)$:

$$\mathsf{H}_{\mathsf{T}}^{\mathsf{A},\mathsf{A}'} := [t_{I,J}]_{I \in [n_1]\times\cdots\times[n_k], J \in [n_{k+1}]\times\cdots\times[n_l]}$$

where $A = [n_1] \times \cdots \times [n_k]$, $A' = [n_{k+1}] \times \cdots \times [n_l]$.

☞ matrix of size $M \times N$ with $M = n_1 \times \cdots \times n_k, N = n_{k+1} \times \cdots \times n_l$.

$$\overbrace{(\mathbb{K}^{n_1} \otimes \cdots \otimes \mathbb{K}^{n_k})}^{E} \otimes \overbrace{(\mathbb{K}^{n_{k+1}} \otimes \cdots \otimes \mathbb{K}^{n_l})}^{F} \quad \sim \quad E \otimes F$$

Flattening of symmetric tensors

For $T \in S^d(\mathbb{K}^n)$,

- **Flattening**, **matricisation**, **Catalecticant**, **hankel** in degree $(k, d-k)$:

$$\mathrm{H}_{\mathrm{T}}^{\mathrm{k,d-k}} := [\langle T, \boldsymbol{x}^{\alpha+\beta}\rangle_d]_{|\alpha|=k,|\beta|=d-k} = [t_{\alpha+\beta}]_{|\alpha|=k,|\beta|=d-k}$$

  $H_T^{k,d-k}$ is also called the **moment** matrix of T.

- **Hankel operator**:

$$\begin{array}{rcl} H_T^{k,d-k} : S^{d-k}(\mathbb{K}^n) & \to & S^k(\mathbb{K}^n)^\star \\ b & \mapsto & \langle T, b\,\cdot\rangle_d = b \star T^* \end{array}$$

- For $A \subset S^k(\mathbb{K}^k), A' \subset S^{d-k}(\mathbb{K}^n)$, $H_T^{A,A'} = [\langle T, a\,a'\rangle_d]_{a\in A, a'\in A'}$.

Example: For $T = x_0^3 + 6x_0^2x_1 + 9x_0x_1^2 + 5x_1^3$,

$$H_T^{1,2} = \begin{array}{c} \\ x_0 \\ x_1 \end{array} \begin{array}{c} \begin{matrix} x_0^2 & x_0x_1 & x_1^2 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} \end{array}.$$

Its kernel is spanned by $[1, 1, -1]$ coresponding to $x_0^2 + x_0x_1 - x_1^2$.

For $T = \sum_{i=1}^{r} \omega_i (\xi_i, \boldsymbol{x})^d$ or $\lambda = \sum_{i=1}^{r} \omega_i \delta_{\xi_i}$ $(\star)$, let

- $I_\Xi := \{p \in \mathbb{K}[\boldsymbol{x}]$ s.t. $p(\xi_i) = 0\}$ be the defining ideal of $\Xi = \{\xi_1, \ldots, \xi_r\}$
- $\mathcal{A}_\Xi := \mathbb{K}[\boldsymbol{x}]/I_\Xi$ the quotient algebra by I_Ξ, of dimension r.

Theorem

Let $A = \{a_1, \ldots, a_s\}, A' = \{a'_1, \ldots, a'_t\}$, and $H = H_T^{A,A'}$ be a **flattening** *of T as $(\star)$.*

$$H_T^{A,A'} = V_{A,\Xi} \Delta_\omega V_{A',\Xi}^t$$

where $V_{A,\Xi} = \begin{bmatrix} a_1(\xi_1) & \cdots & a_1(\xi_r) \\ \vdots & & \vdots \\ a_s(\xi_1) & \cdots & a_s(\xi_r) \end{bmatrix}$ is the **Vandermonde** matrix A, Ξ.

If A and A' contain a basis of $\mathcal{A}_\Xi$, then

- $\ker H = I_\Xi \cap \langle A' \rangle$
- $\operatorname{im} H = (I_\Xi^\perp)_{|\langle A \rangle} = \operatorname{im} V_{A,\Xi}$ *where* $I_\Xi^\perp = \{\lambda \in \mathbb{K}[\boldsymbol{x}]^* \mid \forall p \in I_\Xi, \langle \lambda, p \rangle = 0\} = \mathcal{A}_\Xi^* = \langle \delta_{\xi_1}, \ldots, \delta_{\xi_r} \rangle$.

Example: For $T = x_0^3 + 6x_0^2 x_1 + 9x_0 x_1^2 + 5x_1^3$,

$H_T^{2,1} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \xi_1 & \xi_2 \\ \xi_1^2 & \xi_2^2 \end{bmatrix} \operatorname{diag}(\omega_1, \omega_2) \begin{bmatrix} 1 & \xi_1 \\ 1 & \xi_2 \end{bmatrix}$ with ξ_i roots of $X^2 - X - 1 = 0$ for $X = \frac{x_1}{x_0}$.

The roots by eigencomputation

Hypothesis: $\mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\xi_1, \ldots, \xi_r\} \Leftrightarrow \mathcal{A} = \mathbb{K}[\mathrm{x}]/I$ Artinian (i.e. $\dim_{\mathbb{K}} \mathcal{A} < \infty$).

$$\begin{array}{rcl} \mathcal{M}_a : \mathcal{A} & \to & \mathcal{A} \\ u & \mapsto & a\,u \end{array} \qquad \begin{array}{rcl} \mathcal{M}_a^{\mathsf{t}} : \mathcal{A}^* & \to & \mathcal{A}^* \\ \Lambda & \mapsto & a \star \Lambda = \Lambda \circ \mathcal{M}_a \end{array}$$

Theorem

- *The eigenvalues of* $\mathcal{M}_a$ *are* $\{a(\xi_1), \ldots, a(\xi_r)\}$.
- *The eigenvectors of all* $(\mathcal{M}_a^{\mathsf{t}})_{a \in \mathcal{A}}$ *are (up to a scalar)* $\mathrm{e}_{\xi_i} : p \mapsto p(\xi_i)$.

Proposition

If the roots are **simple**,

- *the operators* $\mathcal{M}_a$ *are diagonalizable,*
- *their common eigenvectors are, up to a scalar,* **interpolation polynomials** u_i *at the roots and idempotent in* $\mathcal{A}$.

Affine setting ("$x_0 = 1$") for homogeneous forms.

☞ $B \subset A$, $B' \subset A'$ are bases of $\mathcal{A}_{\Xi}$ iff $H_0 = H_T^{B',B}$ is invertible.

Assume that $x_i \cdot B \subset A$, let $H_i = H_T^{B',x_i B}$.

☞ $M_i = H_0^{-1} H_i$ is the multiplication by x_i in B modulo I_{Ξ}

Example: For $H_T^{1,2} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}, H_0 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, H_1 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$,

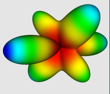
$M_1 = H_0^{-1} H_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is the multiplication by x in $B = \{1, x\}$ mod. $x^2 - x - 1$.

☞ $\exists\, E, F$ invertible such that

$$\mathrm{H_i} = \mathrm{E}\,\mathrm{diag}(\xi_{1,i}, \ldots, \xi_{r,i})\,\mathrm{F} \Rightarrow \textbf{joint diagonalisation} \text{ of } H_0^{-1} H_i.$$

☞ The common **eigenvectors** of M_i^t are (up to a scalar) the vectors $[B(\xi_i)]$, $i = 1, \ldots, r$.

Symmetric tensor decomposition



$$
\begin{aligned}
T &= (x_0+3\,x_1-x_2)^4+(x_0+x_1+x_2)^4-3\,(x_0+2\,x_1+2\,x_2)^4\\
&= -{x_0}^4-8\,{x_0}^3x_1-24\,{x_0}^3x_2-60\,{x_0}^2{x_2}^2-168\,{x_0}^2x_1x_2-12\,{x_0}^2{x_1}^2\\
&\quad -96\,x_0{x_2}^3-240\,x_0x_1x_2^2-384\,x_0x_1^2x_2+16\,x_0{x_1}^3-46\,{x_2}^4-200\,x_1x_2^3\\
&\quad -228\,{x_1}^2{x_2}^2-296\,x_1^3x_2+34\,{x_1}^4\\
\langle T,p\rangle_4 &= \langle T^*|p\rangle \text{ where } T^*=\mathbf{e}_{(1,3,-1)}+\mathbf{e}_{(1,1,1)}-3\,\mathbf{e}_{(1,2,2)} \quad (\textit{by apolarity})
\end{aligned}
$$

$H_T^{2,2} :=$

$$
\begin{bmatrix}
-1 & -2 & -6 & -2 & -14 & -10\\
-2 & -2 & -14 & 4 & -32 & -20\\
-6 & -14 & -10 & -32 & -20 & -24\\
-2 & 4 & -32 & 34 & -74 & -38\\
-14 & -32 & -20 & -74 & -38 & -50\\
-10 & -20 & -24 & -38 & -50 & -46
\end{bmatrix}
$$

For $B' = \{x_0, x_1, x_2\}$,

$$
H_T^{B,x_0 B'} = \begin{bmatrix} -1 & -2 & -6\\ -2 & -2 & -14\\ -6 & -14 & -10 \end{bmatrix}
$$

$$
H_T^{B,x_1 B'} = \begin{bmatrix} -2 & -2 & -14\\ -2 & 4 & -32\\ -14 & -32 & -20 \end{bmatrix}
$$

$$
H_T^{B,x_2 B'} = \begin{bmatrix} -6 & -14 & -10\\ -14 & -32 & -20\\ -10 & -20 & -24 \end{bmatrix}
$$

- The matrix of multiplication by $x_2 x_0^{-1}$ in $x_0\, B' = \{x_0^2, x_0x_1, x_0x_2\}$ is

$$M_{\mathbf{2}} = (H_T^{B, x_{\mathbf{0}} B'})^{-\mathbf{1}} H_T^{B, x_{\mathbf{2}} B'} = \begin{bmatrix} 0 & -2 & -2 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{5}{2} & \frac{3}{2} \end{bmatrix}.$$

- Its eigenvalues are $[-1, 1, 2]$ and the eigenvectors:

$$V := \begin{bmatrix} 0 & -2 & -1 \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

  that is the polynomials

$$V(\boldsymbol{x}) = \left[\ \tfrac{1}{2}\,x_1 - \tfrac{1}{2}\,x_2 \quad -2\,x_0 + \tfrac{3}{4}\,x_1 + \tfrac{1}{4}\,x_2 \quad -x_0 + \tfrac{1}{2}\,x_1 + \tfrac{1}{2}\,x_2\ \right].$$

- We deduce the weights and the frequencies:

$$H_T^{B,V} = \begin{bmatrix} 1 \times 1 & 1 \times 1 & -3 \times 1 \\ 1 \times 3 & 1 \times 1 & -3 \times 2 \\ 1 \times -1 & 1 \times 1 & -3 \times 2 \end{bmatrix}$$

Weights: $1, 1, -3$;

Frequencies: $(1, -1, 3), (1, 1, 1), (1, 2, 2)$.

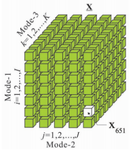
Decomposition:

$$T^* = \mathrm{e}_{(1,3,-1)} + \mathrm{e}_{(1,1,1)} - 3\,\mathrm{e}_{(1,2,2)}$$

$$T = (\mathrm{x}_0 + 3\,\mathrm{x}_1 - \mathrm{x}_2)^4 + (\mathrm{x}_0 + \mathrm{x}_1 + \mathrm{x}_2)^4 - 3\,(\mathrm{x}_0 + 2\,\mathrm{x}_1 + 2\,\mathrm{x}_2)^4$$

Multilinear tensors

$T \in \mathbb{K}^{n_1} \otimes \mathbb{K}^{n_2} \otimes \mathbb{K}^{n_3} \quad \equiv \quad [T_{[k]}]_{k=1}^{n_3}$ pencil of n_3 matrices of size $n_1 \times n_2$.



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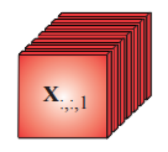


For $T \in \mathbb{K}^{n_1} \otimes \mathbb{K}^{n_2} \otimes \mathbb{K}^{n_3}$ and $r \leq \min\{n_1, n_2\}$

$$T = \sum_{i=1}^{r} \omega_i \boldsymbol{u}_i \otimes \boldsymbol{v}_i \otimes \boldsymbol{w}_i \text{ with } \boldsymbol{u}_i \in \mathbb{K}^{n_1}, \boldsymbol{v}_i \in \mathbb{K}^{n_2}, \boldsymbol{w}_i \in \mathbb{K}^{n_3}$$

iff $\mathrm{T}^* = \sum_{i=1}^{r} \omega_i \, \delta_{\boldsymbol{u}_i} \otimes \delta_{\boldsymbol{v}_i} \otimes \delta_{\boldsymbol{w}_i}$

iff $\mathrm{T}_{[k]} = U \, \mathrm{diag}(w_{i,1}, \ldots, w_{i,r}) V^t \quad$ for $k \in 1{:}n_3$

If $r = n_1 = n_2$ and $T_{[1]}$ of rank r,

- $U =$ matrix of **common eigenvectors** of $M_i = T_{[i]} \, T_{[1]}^{-1}$
- $V^{-t} =$ matrix of **common eigenvectors** of $M_i' = T_{[1]}^{-1} T_{[i]}$.

Approximate decomposition

Objective: find Ξ of smallest size r such that ank

$$H_T^{A',A} \approx V_{A',\Xi}\, \Delta_\omega V_{A,\Xi}^t = P\, N$$

where

- $V_{A,\Xi}$ = **Vandermonde** of A, Ξ of rank $r = |\Xi|$,
- $P = V_{A',\Xi} F \in \mathbb{K}^{s' \times r}$, $N = G\, V_{A,\Xi}^t \in \mathbb{K}^{r \times s}$ with $G \in \mathbb{K}^{r \times r}$ invertible and $F = \Delta_\omega G^{-1}$.

☞ **rank factorisation** with factors of the form $V_{A,\Xi} E$, with E invertible.

☞ $\{V_{A,\Xi}\, G, G \in \mathrm{Gl}(\mathbb{K}^r)\} \equiv I^{\perp}_{|\langle A \rangle}$ is a r linear space of $\boldsymbol{D} := \langle A \rangle^{\star}$

☞ a point of the **Grassmannian** $\mathrm{Gr}^r(\boldsymbol{D})$, in the reduced component $\mathrm{Hilb}^{red}_{r,n}$ of the **Hilbert scheme** $\mathrm{Hilb}_{r,n}$.

Strategy:

- Find the *closest rank-r factorisation* via truncated SVD:

$$H^{[r]} = U^{[r]} S^{[r]} (V^{[r]})^t = P\,N$$

- Find a *closest point* to P (resp. N) on $\mathrm{Hilb}_{r,n}$.
  Given $N = G\, V^t_{A,\Xi} \in \mathbb{K}^{r\times s}$ with G invertible,
  - compute $N_0 \in \mathbb{K}^{r\times r}$ invertible indexed by B, N_i indexed by $x_i \cdot B$
  - compute $M_i = N_0^{-1} N_i$
  - Compute the nearest joint diagonalisation of $[M_1, \ldots, M_n]$

Experimentation (Chuong Luong)

i) Compute (approximations of) the moments $\sigma_\alpha = \int x^\alpha d\mu$ for a measure μ.

ii) Decompose
$T(\mathbf{x}) = \int (1 + (\mathbf{x}, \mathbf{y}))^d d\mu(\mathbf{y}) = \sum_{|\alpha| \le d} \sigma_\alpha \binom{d}{\alpha} \mathbf{x}^\alpha = \sum_{i=1}^r \omega_i (1 + (\xi_i, \mathbf{x}))^d$

Joint Diagonalization

1. using the single diagonalisation of a random combination of the M_i, or
2. by minimization of

$$\min_{E \text{ inv.}} \sum_i \|E M_i E^{-1}\|_{\text{off}}$$

with Jacobi updates $E_{k+1} = (I + X_k) E_k$ and gradient descent. [P. Catalat]

Computing the weights

a) explicit formulae from the joint eigenvectors, or

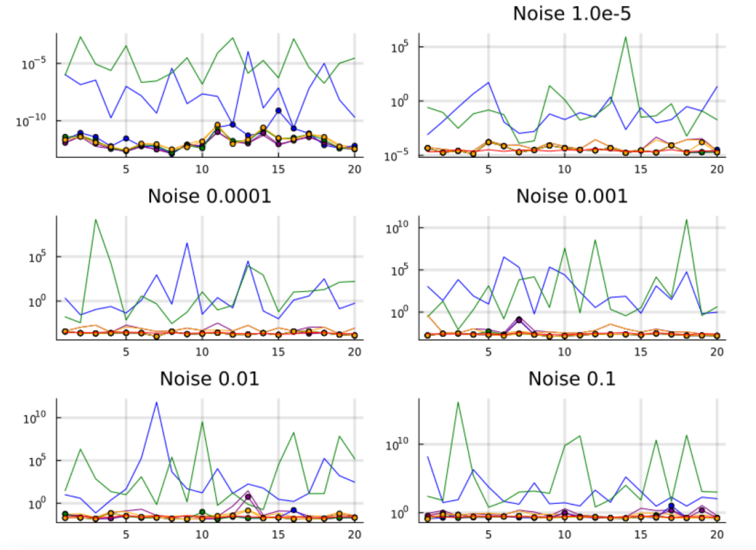
b) solving a Vandermonde system $V_{A,\Xi}\, \omega = B$.

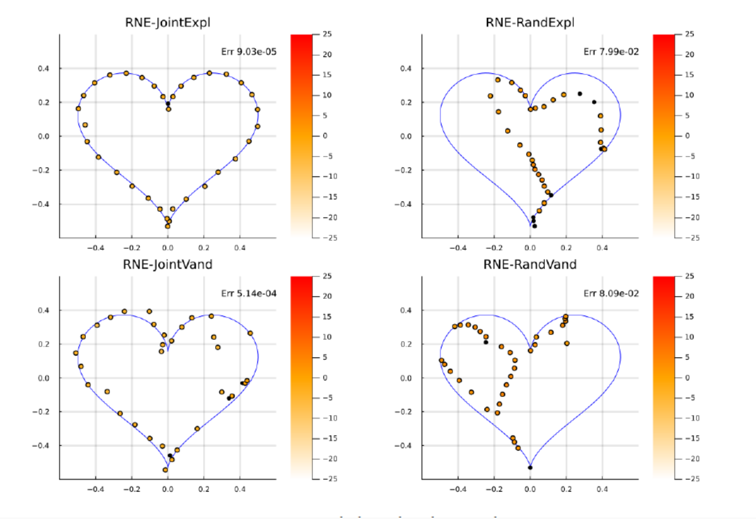
Improving the decomposition

▶ Minimization of $\| T - \sum_i \omega_i (\xi, \mathbf{x})^d \|$ with Riemannian Newton steps (RNE''') and trust-region scheme [R. Khouja]

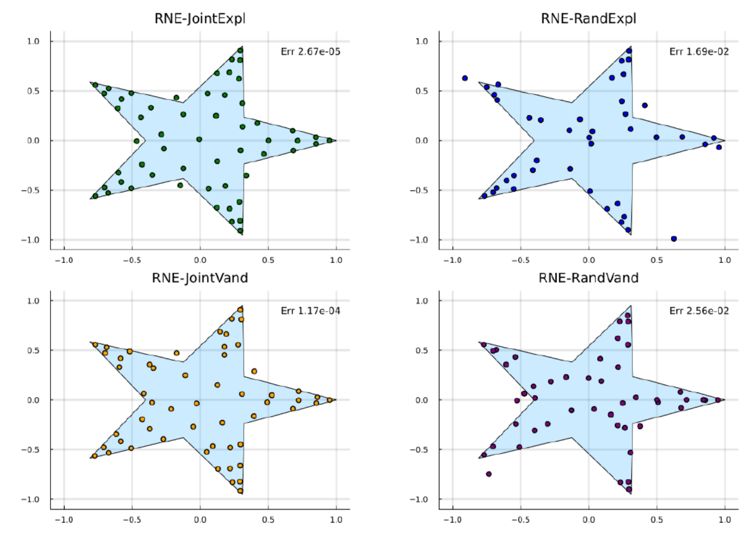
$n = 4, d = 7, r = 9.$

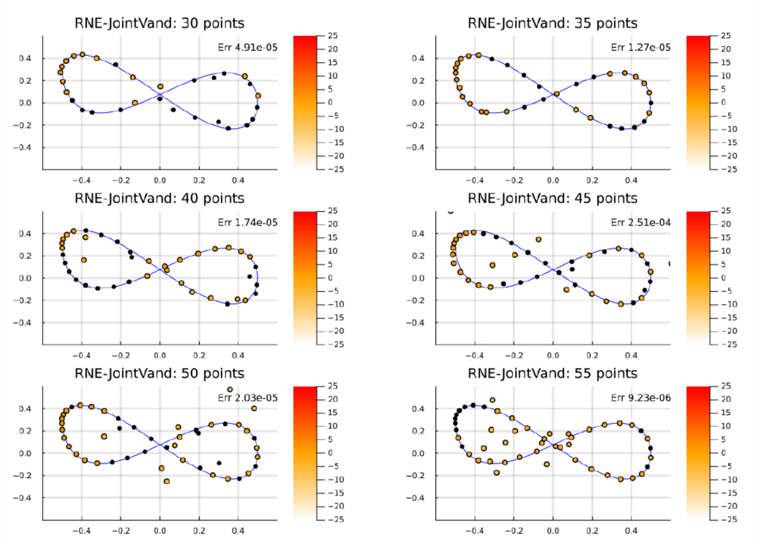
– RandExpl, – JointExpl, – RandVand, – JointVand, – Noise, ○ with RNE.





RNE-JointExpl
Err 9.03e-05
RNE-RandExpl
Err 7.99e-02
RNE-JointVand
Err 5.14e-04
RNE-RandVand
Err 8.09e-02


RNE-JointExpl
Err 2.67e-05
RNE-RandExpl
Err 1.69e-02
RNE-JointVand
Err 1.17e-04
RNE-RandVand
Err 2.56e-02
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0


RNE-JointVand: 30 points
Err 4.91e-05
RNE-JointVand: 35 points
Err 1.27e-05
RNE-JointVand: 40 points
Err 1.74e-05
RNE-JointVand: 45 points
Err 2.51e-04
RNE-JointVand: 50 points
Err 2.03e-05
RNE-JointVand: 55 points
Err 9.23e-06

Thanks for your attention

TENORS

Tensor modEliNg, geOmetRy and optimiSation

Marie Skłodowska-Curie Doctoral Network, 2024-2027

Tensors are nowadays ubiquitous in many domains of applied mathematics, computer science, signal processing, data processing, machine learning and in the emerging area of quantum computing. TENORS aims at fostering cutting-edge research in tensor sciences, stimulating interdisciplinary and intersectoriality knowledge developments between algebraists, geometers, computer scientists, numerical analysts, data analysts, physicists, quantum scientists, and industrial actors facing real-life tensor-based problems.

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