Decoupling multivariate functions using tensors

Mariya Ishteva

November 26, 2024

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A toy example

$x_1 \longrightarrow$ $\longrightarrow y_1$

$$f_1(x_1, x_2) = 54x_1^3 - 54x_1^2x_2 + 8x_1^2 + 18x_1x_2^2 + 16x_1x_2 - 2x_2^3 + 8x_2^2 + 8x_2 + 1,$$

$$f_2(x_1, x_2) = -27x_1^3 + 27x_1^2x_2 - 24x_1^2 - 9x_1x_2^2 - 48x_1x_2 - 15x_1 + x_2^3 - 24x_2^2 - 19x_2 - 3$$

$x_2 \longrightarrow$ $\longrightarrow y_2$

A toy example

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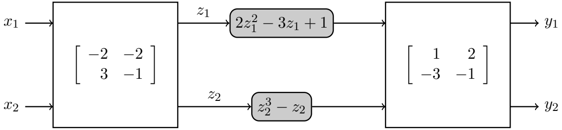
$$\updownarrow$$



Based on joint work with

Philippe Dreesen

Konstantin Usevich

Joppe De Jonghe

Johan Schoukens

Koen Tiels, Maarten Schoukens, David Westwick,
Ivan Markovsky, Gabriel Hollander, Thomas Goossens,
Yassine Zniyed, André de Almeida, ...

Decoupling multivariate functions using tensors

History

Computation

Applications

Waring's problem (1770)

Every natural number can be represented as
the sum of at most 4 squares, 9 cubes, or 19 fourth powers.

$$30 = 5^2 + 2^2 + 1^2$$
$$30 = 3^3 + 1^3 + 1^3 + 1^3$$
$$30 = 2^4 + 1^4 + \ldots + 1^4$$

Hilbert–Waring theorem, 1909:

$$\forall k \in \mathbb{N}, \exists s \in \mathbb{N} : \quad \forall n \in \mathbb{N} : \quad n = \sum_{i \leq s} a_i^k, \quad a_i \in \mathbb{N}, i = 1, \ldots, s.$$

Waring's problem for homogeneous polynomials

Decompose a homogeneous multivariate polynomial $f(x_1, \ldots, x_m)$ of degree d as

$$f(x_1, \ldots, x_m) = \sum_{i=1}^{r} u_i (v_{1i} x_1 + \cdots + v_{mi} x_m)^d;$$

r is called the Waring rank.

Waring's problem for a set of non-homogeneous polynomials

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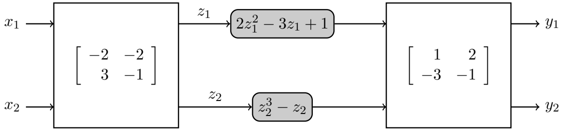

$$\mathbf{f}(\mathbf{x}) = \mathbf{U}\mathbf{g}(\mathbf{V}^T\mathbf{x})$$

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We compute the decomposition by a first-order approach

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then $$\underbrace{\left[\frac{\partial f_i(\mathbf{x})}{\partial x_j}\right]}_{\mathbf{J_f}(\mathbf{x})} = \mathbf{U}\begin{bmatrix} g_1'(\mathbf{v}_1^T\mathbf{x}) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & g_r'(\mathbf{v}_r^T\mathbf{x}) \end{bmatrix}\mathbf{V}^T.$$

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- Collect Jacobian matrices $\mathbf{J}_\mathbf{f}^{(1)}, \mathbf{J}_\mathbf{f}^{(2)}, \mathbf{J}_\mathbf{f}^{(3)}, \mathbf{J}_\mathbf{f}^{(4)}, \mathbf{J}_\mathbf{f}^{(5)}, \ldots$ and diagonalize them simultaneously

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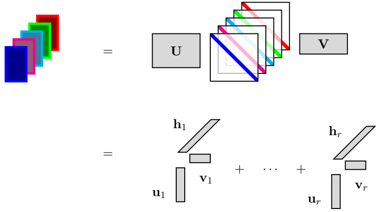
  Tool: Canonical Polyadic Decomposition (CPD)

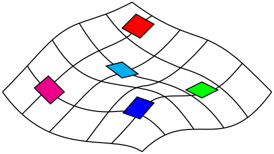


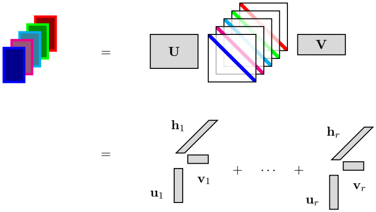
Algorithm

1. Construct tensor of Jacobians
   $\mathcal{J}_{\mathbf{f}} = \left\{ \mathbf{J}_{\mathbf{f}}^{(1)}, \mathbf{J}_{\mathbf{f}}^{(2)}, \mathbf{J}_{\mathbf{f}}^{(3)}, \mathbf{J}_{\mathbf{f}}^{(4)}, \mathbf{J}_{\mathbf{f}}^{(5)}, \ldots \right\}$

2. CPD of $\mathcal{J}_{\mathbf{f}}$ gives $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{H}$

3. Retrieve coefficients of $g_i(\cdot)$ from $\mathbf{y}^{(k)} = \mathbf{U}\left[g_i(\mathbf{v}_i^T \mathbf{x}^{(k)})\right]$
   (solving linear system)





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  $\rightarrow$ ParaTuck-Z decomposition

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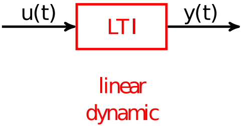
- System identification (2)
- Neural networks

Block-oriented models provide a good trade-off between simplicity and descriptive power



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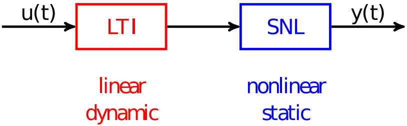

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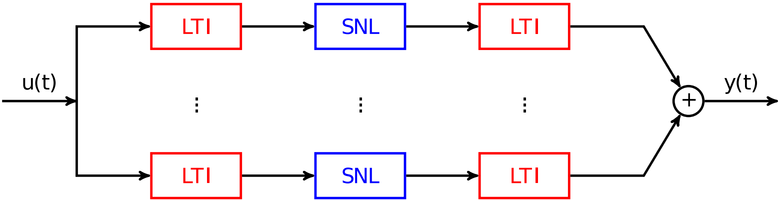
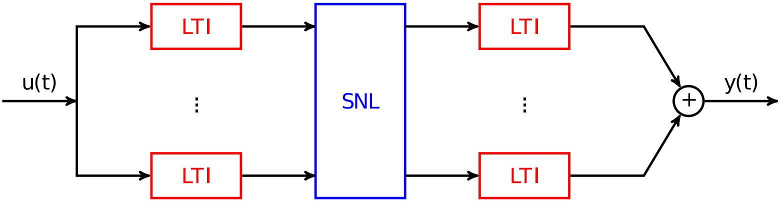


Block-oriented models: parameter estimation



Step 1: estimate the parameters of the LTI blocks
Step 2: fit a multivariate polynomial
Step 3: decouple the multivariate polynomial

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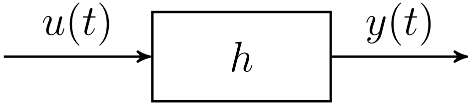
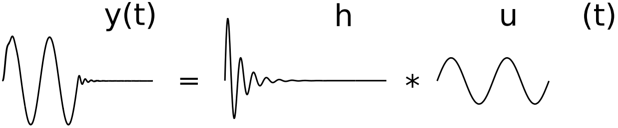
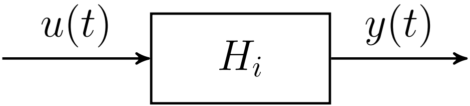
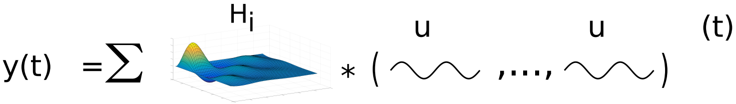
The impulse response completely characterizes *linear* dynamical systems





The Volterra kernels completely characterize *nonlinear* dynamical systems





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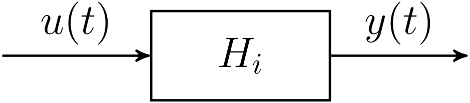
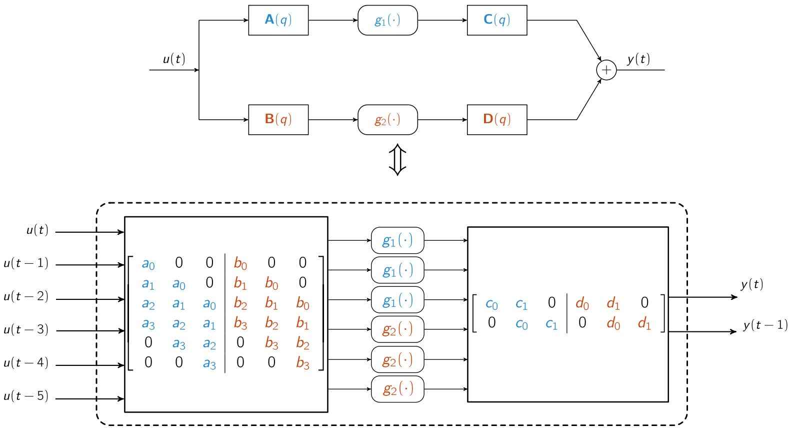


Volterra series are polynomials
of time-shifted input signals

$$y(t) = \sum_i \sum_{\tau_1,\ldots,\tau_i} \underbrace{H_i(\tau_1,\ldots,\tau_i)}_{\text{kernels}} \underbrace{u(t-\tau_1)\cdots u(t-\tau_i)}_{\text{time-shifted inputs}}$$

Decoupling Volterra representations



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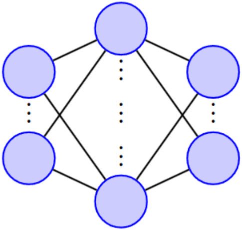
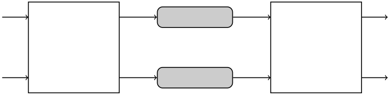
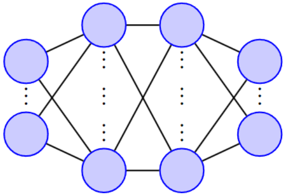
Neural networks with one layer are decoupled functions but can be compressed using flexible activation functions





Neural networks with two layers can be compressed to one layer with flexible activation functions



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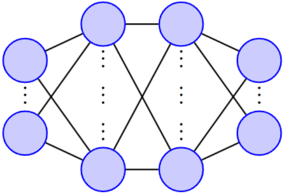
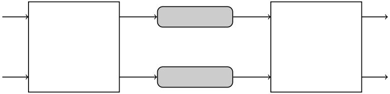
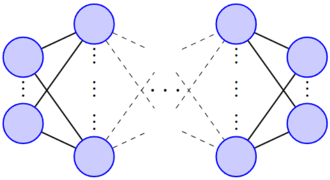




Neural networks with multiple layers can be compressed to one layer with flexible activation functions



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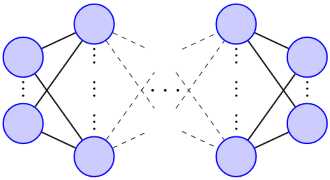
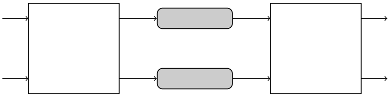
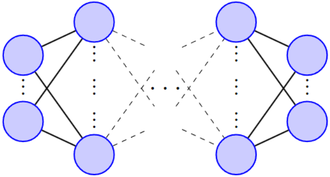
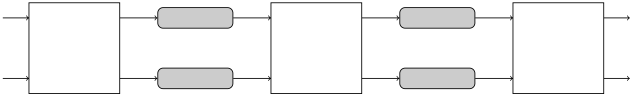




Neural networks with multiple layers can be compressed to two layers with flexible activation functions using ParaTuck-2





- Multilayer Tensor-based Neural Network Compression with Flexible Activation Functions
- A lifting approach to ParaTuck-2 tensor decompositions

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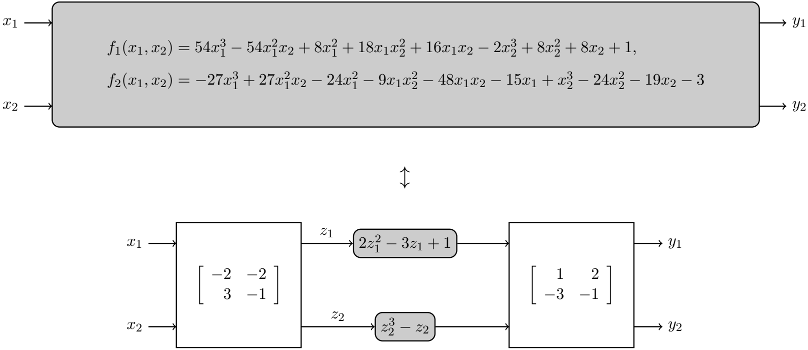

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